PAW6

Formalizm lagranżowski

Algorytm

1° $q, \dot q \in R^n$ $\quad t \in [t_0, t_1]$ współ. uogólnione

2° $L(q,\dot q) = K(q,\dot q) - V(q)$

3° 2ND $\quad I(q(\cdot)) = \int_{t_0}^{t_1} L(q,\dot q)\, dt \to extr$

4° REL $\quad \frac{d}{dt}\frac{\partial L}{\partial \dot q} - \frac{\partial L}{\partial q} = 0 \ (F)$

Postać energii kinetycznej

$$K(q,\dot q) = \frac{1}{2}\dot q^T Q(q)\dot q$$

$$\underbrace{\frac{\partial^2 L}{\partial \dot q^2}}_{Q(q)}\ddot q + \underbrace{\frac{\partial^2 L}{\partial q \partial \dot q}}_{C(q,\dot q)}\dot q - \underbrace{\frac{\partial L}{\partial q}}_{D(q)} = F$$

$+R\dot q$ siły tarcia

macierz inercji; macierz sił odśrodkowych i Coriolisa; wektor sił potencjalnych; wektor sił niepotencjalnych

---

np. belka i kulka

$q = \begin{pmatrix} r \\ \varphi \end{pmatrix}$

$$L = K - V = \frac{1}{2} I \dot\varphi^2 + \frac{1}{2} m \dot r^2 + \frac{1}{2} m r^2 \dot\varphi^2 - mgr s\varphi$$

$J = N\cdot m = \frac{kg\cdot m^2}{s^2} \leftarrow$ $kg\cdot m^2 \ \frac{1}{s^2}$ $\quad kg \left(\frac{m}{s}\right)^2$ $\quad kg\ m^2\ \frac{1}{s^2}$ $\quad kg\ \frac{m}{s^2}\ m$

$[N] = \left[\frac{kg\cdot m}{s^2}\right]$ $\quad m\ddot r - mr\dot\varphi^2 + mg s\varphi = 0$ $\quad$ odśrodkowe

$[N\cdot m] = \left[\frac{kg\cdot m^2}{s^2}\right]$ $\quad (I + mr^2)\ddot\varphi + 2mr\dot r\dot\varphi + mgr c\varphi = 0 \ (\tau$ - napęd belki$)$

bezwładność; Coriolis; reszta po lewej potencjalne

rozwiązanie

$\boxed{\text{trajektoria} \begin{cases} r(t) \\ \varphi(t) \end{cases}}$

np. wahadło Furuty (2DOF)

PZU6

$\boxed{1^\circ}$ $q = \begin{pmatrix} \theta \\ \varphi \end{pmatrix}$ $\dot{q} = \begin{pmatrix} \dot\theta \\ \dot\varphi \end{pmatrix} \in R^2$

$\boxed{2^\circ}$ $L = K - V \leftarrow$ potrzebujemy położeń i prędkości wszystkich mas

Z „geometrii" układu

$$\boxed{\begin{cases} x = (a + b\sin\varphi)\cos\theta \\ y = (a + b\sin\varphi)\sin\theta \\ z = l - b\cos\varphi \end{cases}}$$

Stąd $\dot x, \dot y, \dot z$ i $v^2 = (a + b\sin\varphi)^2\dot\theta^2 + b^2\dot\varphi^2$

$\boxed{K = \frac{1}{2}mv^2}$ zaś $\boxed{V = -m(\bar r, \bar g)}$ $\bar r = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\bar g = \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix}$

$$\boxed{L = K - V = \frac{1}{2}m(a + b s\varphi)^2\dot\theta^2 + \frac{1}{2}mb^2\dot\varphi^2 + mgb\,c\varphi}$$

RE.-L. $\frac{d}{dt}\frac{\partial L}{\partial \dot q} - \frac{\partial L}{\partial q} = 0$

$$\begin{cases} (1) & \frac{d}{dt}\frac{\partial L}{\partial \dot\theta} - \frac{\partial L}{\partial \theta} = 0 \\ (2) & \frac{d}{dt}\frac{\partial L}{\partial \dot\varphi} - \frac{\partial L}{\partial \varphi} = 0 \end{cases}$$

(1) $\frac{\partial L}{\partial \dot\theta} = m(a + b s\varphi)^2\dot\theta$ $\frac{\partial L}{\partial \theta} = 0$

(2) $\frac{\partial L}{\partial \dot\varphi} = mb^2\dot\varphi^2$ $\frac{\partial L}{\partial \varphi} = m(a + b s\varphi)\dot\theta^2 b\cos\varphi - mgb s\varphi$

$$\begin{cases} m(a + b s\varphi)^2\dot\theta = \text{const} \\ mb^2\ddot\varphi - m(a b s\varphi)\dot\theta^2 b\,c\varphi + mgb s\varphi = 0 \end{cases}$$

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = k\begin{pmatrix}\dot{\varphi}_1\\ \dot{\varphi}_2\end{pmatrix}$$

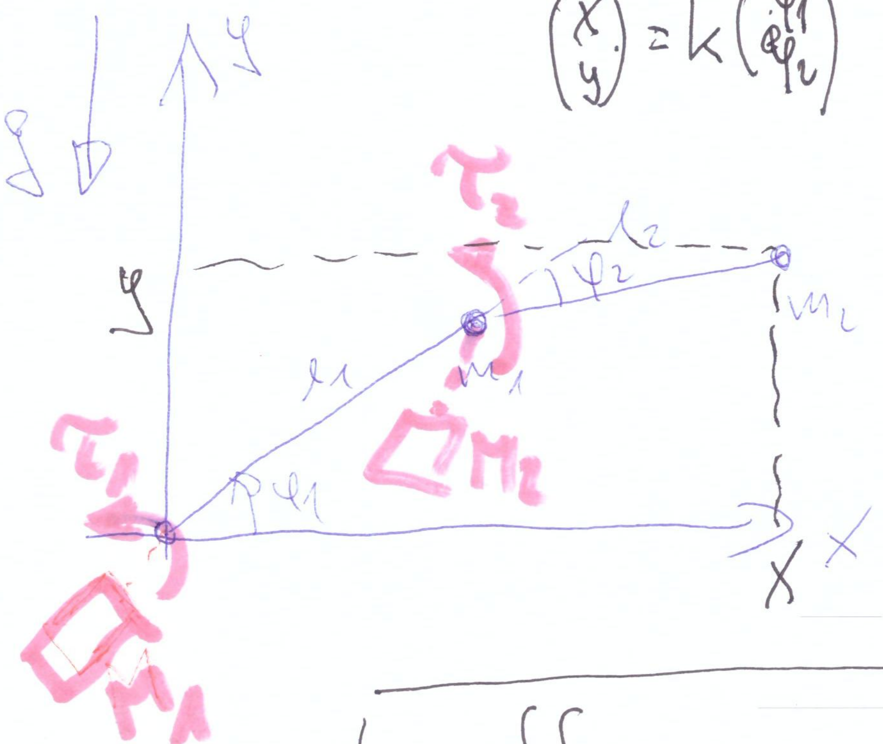
$\frac{1}{2} m v^2$ $\quad \dot{q}$

$\frac{1}{2}\dot{q}^T M \dot{q}$

$Q(q)$

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = J\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}\cdot\begin{pmatrix}\dot{\varphi}_1\\ \dot{\varphi}_2\end{pmatrix}$$

$$\begin{pmatrix}x\\ y\end{pmatrix} \quad \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}$$



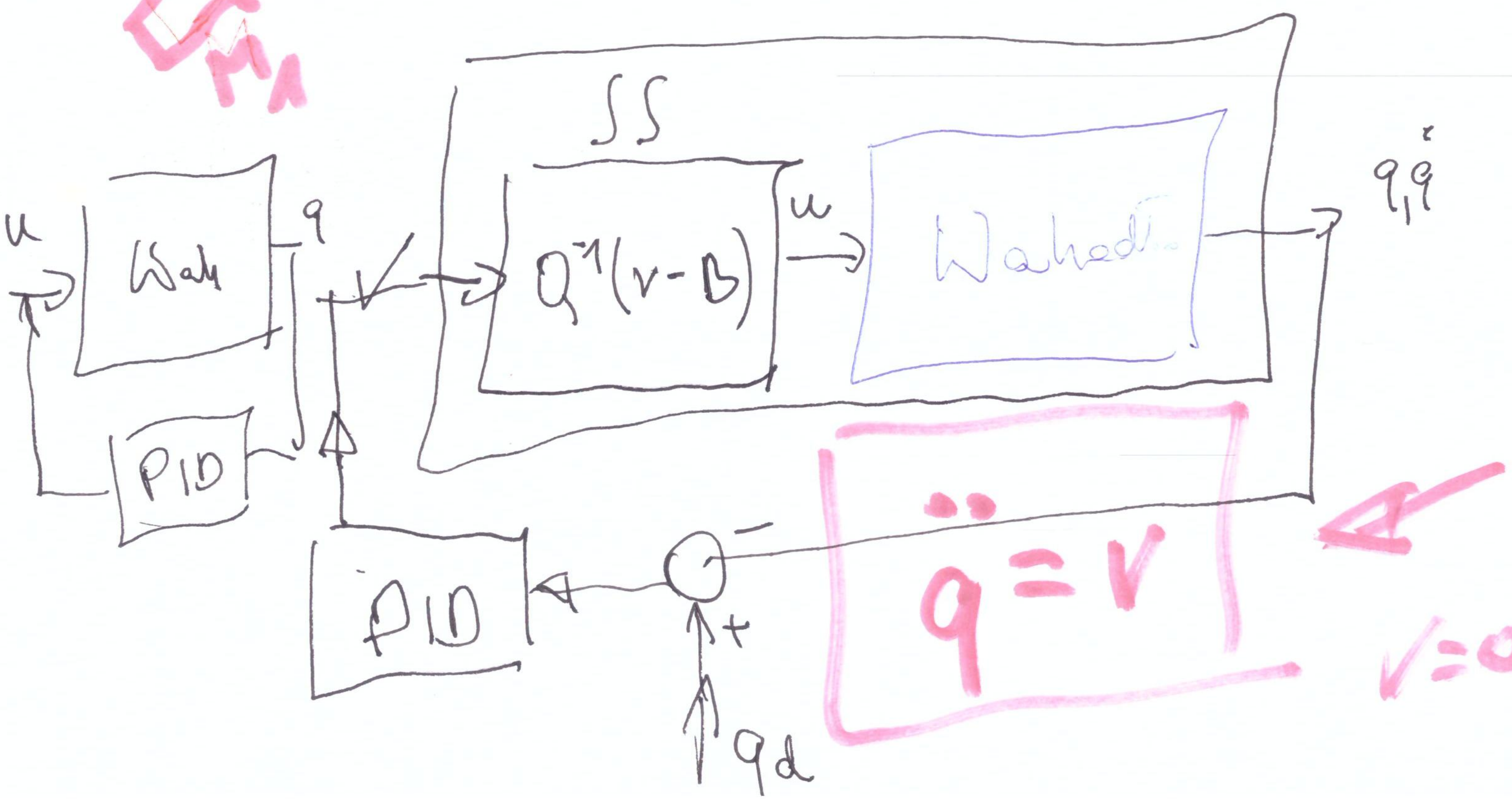
$$Q(q)\cdot\ddot{q} + \underbrace{\boxed{C(q\,\dot{q})\,\dot{q} + D(q)}}_{B(q\,\dot{q})} = u$$

$$C\ddot{q} + B = u$$

$$C\ddot{q} = u - B \quad / C^{-1}$$

"Q"

$$\ddot{q} = C^{-1}(u - B)$$



Lagranżian

$L = \dot q^T Q \dot q$

Postać równań

$Q\ddot q + C\dot q = 0$

$\Downarrow$

$Q\ddot q = -C\dot q$

Własności

$\dot Q = C + C^T$

P3W6

$$\frac{\partial L}{\partial t} = \underbrace{\left(\frac{1}{2}\ddot q^T Q \dot q\right)^T + \frac{1}{2}\dot q^T Q \ddot q}_{\dot q^T \underbrace{(Q^T+Q)}_{2Q}\ddot q} + \frac{1}{2}\dot q^T \dot Q \dot q$$

$(\ )^T$ bo to liczby

$2Q$ – Q macierz symetryczna

$$\frac{\partial L}{\partial t} = \dot q^T Q \ddot q + \frac{1}{2}\dot q \dot Q \dot q = -\dot q^T C \dot q + \frac{1}{2}\dot q^T \dot Q \dot q =$$

$$= \frac{1}{2}\dot q^T(\dot Q - 2C)\dot q = \frac{1}{2}\dot q^T \underbrace{(C + C^T - 2C)}_{C^T - C}\dot q = \frac{1}{2}\dot q(C^T - C)\dot q$$

co to $C^T - C$? oznaczmy jako M

$M = C^T - C \qquad M^T = (C^T - C)^T = C - C^T = -(C^T - C) = -M$

$M^T = -M$ – macierz skośnie symetryczna

jej własności:

forma kwadratowa zbudowana na macierzy skośniesymetrycznej jest równa 0

więc $\boxed{\frac{\partial L}{\partial t} = \frac{1}{2}\dot q^T (C^T - C)\dot q = 0}$

Lagranżian jest stały na trajektoriach

$$L = \int_{t_0}^{t_1} \|\dot q\|\, dt \to \min$$ ← długości krzywej

$$\sqrt{(\dot q, \dot q)}$$

metryka kartezjańska

$$\sqrt{(\dot q\ \dot q)} = \sqrt{\dot q^T I \dot q}$$

$$\sqrt{\dot q^T \underline{Q(q)}\, \dot q}$$

$$\sqrt{\dot q^T I \dot q}$$ ← szczególny przypadek

$$\sqrt{\dot q^T Q(q)\, \dot q}$$

$$Q(q) = I$$

metryka Riemanna

$$q = \begin{pmatrix}\varphi\\ \theta\end{pmatrix} \qquad \begin{pmatrix}\varphi(t)\\ \theta(t)\end{pmatrix} \to \begin{pmatrix}x(t)\\ y(t)\\ z(t)\end{pmatrix}$$

$$\begin{pmatrix}\dot x\\ \dot y\\ \dot z\end{pmatrix} = DF\begin{pmatrix}\varphi\\ \theta\end{pmatrix}\begin{pmatrix}\dot \varphi\\ \dot \theta\end{pmatrix} \qquad DF\begin{pmatrix}\varphi\\ \theta\end{pmatrix} = \begin{bmatrix}\frac{\partial F_1}{\partial \varphi} & \frac{\partial F_1}{\partial \theta}\\ \frac{\partial F_2}{\partial \varphi} & \frac{\partial F_2}{\partial \theta}\\ \frac{\partial F_3}{\partial \varphi} & \frac{\partial F_3}{\partial \theta}\end{bmatrix}$$

$$DF = \begin{bmatrix} -\sin\theta \sin\varphi & \cos\theta\cos\varphi \\ \sin\theta\cos\varphi & \cos\theta\sin\varphi \\ 0 & -\sin\theta \end{bmatrix}$$

$v_1 = DF\, u_1 \qquad v_1^T = u_1^T DF^T$

$v_2 = DF\, u_2$

$$(v_1\ v_2)_I = v_1^T v_2 = \underbrace{u_1^T\, DF^T DF\, u_2}_{\downarrow Q} = u_1^T Q\, u_2 = \underline{(u_1\ u_2)_Q}$$

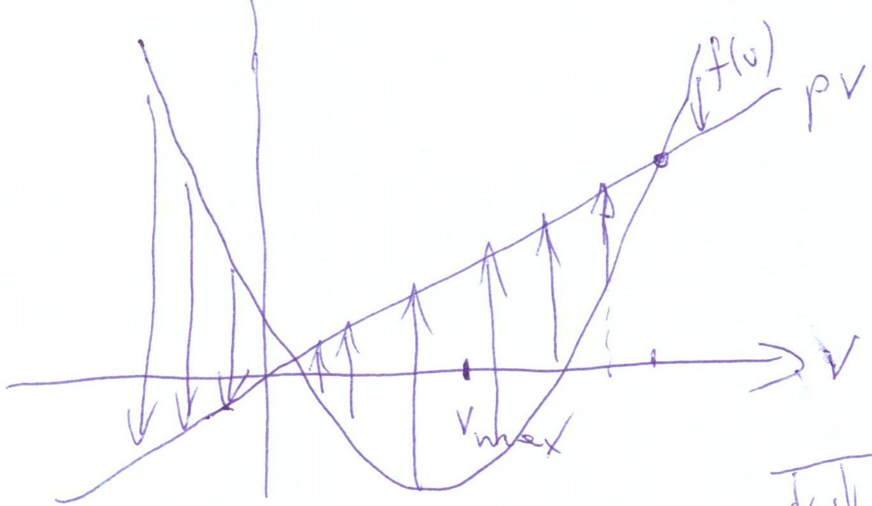
$$(u_{11}\ u_{12}) \begin{bmatrix} \sin^2\theta & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} u_{11} \\ u_{12} \end{pmatrix} \qquad (v_{11}\ v_{12}) \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} = v_{11}^2 + v_{12}^2$$

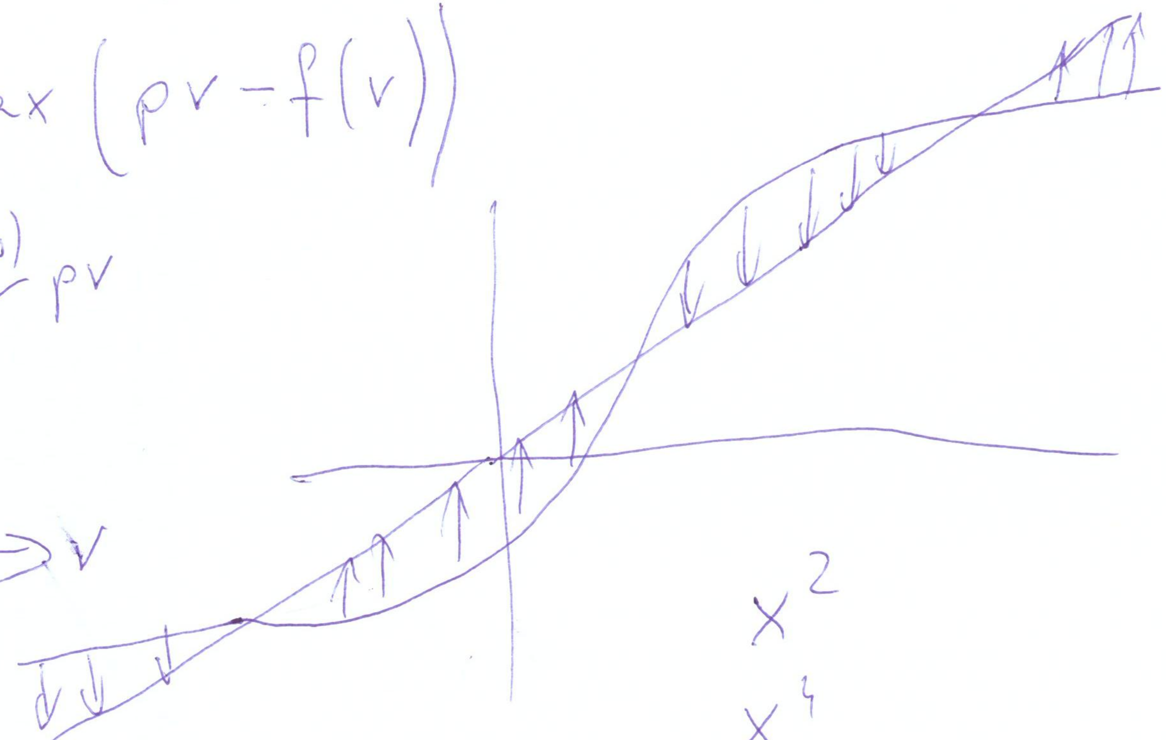
$\theta = \frac{\pi}{2}$ $\qquad \theta = \frac{\pi}{3}$ $\qquad \theta = 0$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \bar{I} \qquad \begin{bmatrix} \frac{3}{4} & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$f: R \to R \qquad y = f(v)$

$F(p) = \max_{v} \left(pv - f(v) \right)$





x^2

x^4

$L(\underset{\text{param}}{q}, \dot q) \quad \dot q = v \qquad \underbrace{\left(p^T v - L_q(v) \right)}_{v \ \max} \qquad v \in R^n$