Y1W3

Mechanika newtonowska

przestrzeń wektorowa

rachunek różniczkowy w $\boxed{R^n}$

Prawo powszechnego ciążenia

Prawa ruchu (Zasady dynamiki)

versus

Zasada zachowania : pędu
momentu pędu
energii

Mechanika lagranżowska

przestrzeń Banacha

rachunek wariacyjny w $\boxed{\mathcal{X}}$

Przestrzeń Banacha – uogólnienie wektorowej

- liniowa $\boxed{x, y \in \mathcal{X} \quad \alpha, \beta \in R \quad \alpha x + \beta y \in \mathcal{X}}$

- unormowana $\boxed{\|\cdot\| : \mathcal{X} \to R}$

$\|x\| \geq 0 \qquad \|x\| = 0 \Leftrightarrow x = 0$

$\|\alpha x\| = |\alpha| \, \|x\|$

$\|x + y\| \leq \|x\| + \|y\|$

- zupełna $\boxed{x_1, \dots x_n, \dots \in \mathcal{X}}$ + ciąg zbieżny

$$\boxed{\lim_{n \to \infty} x_n \in \mathcal{X}}$$

P2W3

Rachunek wariacyjny

Pochodna

- klasycznie w punkcie x_0 $\quad f: R^n \to R^n$

$$\boxed{f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}}$$

$$\boxed{\lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0) - f'(x_0)\Delta x}{\Delta x} = 0}$$

- Frécheta $\quad f: \mathcal{X} \to R$

$$\boxed{\lim_{\|v\| \to 0} \frac{\|f(x+v) - f(x) - \boxed{Df(x)}v\|}{\|v\|} = 0}$$

- Gâteaux (kierunkowa)

$$\boxed{Df(x)v = \frac{d}{d\alpha}\Big|_{\alpha=0} f(x+\alpha v)}$$

Dla $f: R^n \to R$

$$\boxed{Df(x) = \frac{\partial f(x)}{\partial x}}$$

Funkcja

z „liczb" w „liczby"

$$f: R \to R$$

$$\boxed{f(x),\ x \in R}$$

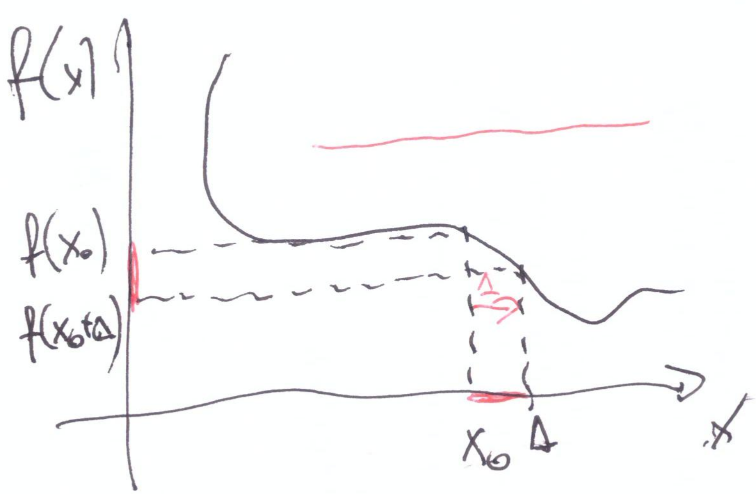
Funkcjonał

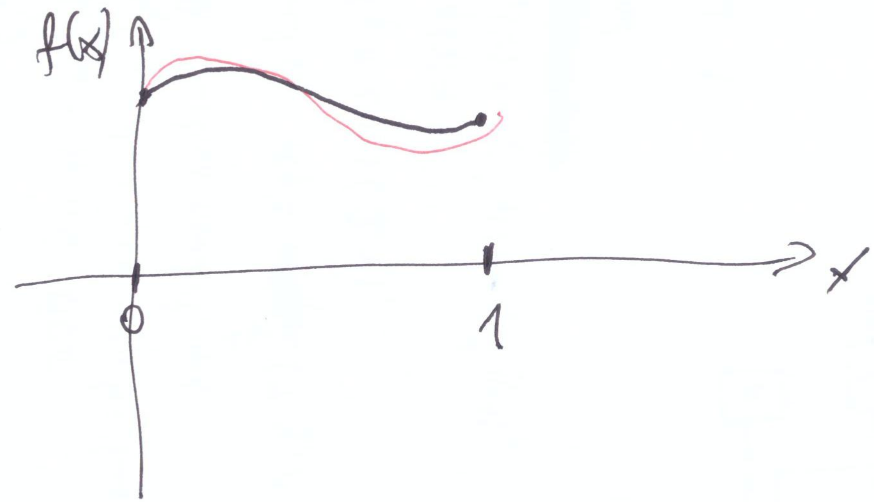
z „funkcji" w „liczby"

$$f: \mathcal{X} \to R$$

$$\mathcal{X} = \{x: [t_0, t_1] \to R^n\}$$

$$\boxed{f(x(\cdot)) \quad x(\cdot) \in \mathcal{X}}$$





$$f(\cdot): [0,1] \to \cancel{\emptyset}?\ \mathbb{R}$$

$$f(\sin(\cdot)) = ?$$

Przykład 3.5.1

$$f(x) = x^T Q x \quad x \in R^n \quad Q_{n\times n}$$

$$Df(x)v = \frac{d}{d\alpha}\Big|_{\alpha=0} (x+\alpha v)^T Q (x+\alpha v) =$$

$$= v^T Q (x+\alpha v) + (x+\alpha v)^T Q v \Big|_{\alpha=0} =$$

$$= (v^T Q x)^T + x^T Q v = x^T Q^T v + x^T Q v =$$

$$= x^T (Q^T + Q) v$$

$$(v^T Q x)^T \quad (1)$$

$$x^T Q^T v$$

$$x^T Q v \quad (2)$$

$$\Rightarrow x^T (Q^T + Q) v \quad (1)+(2)$$

Funkcja

$\boxed{x_1(t) = t^2 + 3}$ $\boxed{x_2(t) = \sin t}$

$\boxed{t_1 = 0}$

$x_1(t_1) = 3 \qquad x_2(t_1) = 0$

$\boxed{t_2 = \frac{\pi}{6}}$

$x_1(t_2) = \frac{\pi^2}{36} + 3 \qquad x_2(t_2) = \frac{1}{2}$

$\boxed{t_3 = 1}$

$x_1(t_3) = 4 \qquad x_2(t_3) \cong 0{,}8415$

Funkcjonał (przy funkcji $\in R^2$)

$\boxed{\bar{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}}$

$\boxed{f_1(x(\cdot)) = \int_0^1 x_2(t)\dot{x}_1(t)\,dt}$ $\boxed{f_2(x(\cdot)) = \int_0^1 \sqrt{\dot{x}_1^2(t) + \dot{x}_2^2(t)}\,dt}$

$\boxed{\bar{x}_1(t) = \begin{pmatrix} t \\ 0 \end{pmatrix}}$

$f_1(\bar{x}_1(\cdot)) = \int_0^1 0\,dt = 0 \qquad f_2(\bar{x}_1(\cdot)) = \int_0^1 1\,dt = 1$

$\boxed{\bar{x}_2(t) = \begin{pmatrix} t \\ t \end{pmatrix}}$

$f_1(\bar{x}_2(\cdot)) = \int_0^1 t\,dt = \frac{1}{2}t^2\Big|_0^1 = \frac{1}{2} \qquad f_2(\bar{x}_2(\cdot)) = \int_0^1 \sqrt{2}\,dt = \sqrt{2}\,t\Big|_0^1 = \sqrt{2}$

$\boxed{\bar{x}_3(t) = \begin{pmatrix} t \\ t^2 + 3 \end{pmatrix}}$

$f_1(\bar{x}_3(\cdot)) = \int_0^1 (t^2 + 3)\,dt = 3\frac{1}{3} \qquad f_2(\bar{x}_3(\cdot)) = \int_0^1 (1 + 4t^2)^{\frac{1}{2}}\,dt = 1{,}4789$

$\boxed{\bar{x}_4(t) = \begin{pmatrix} t \\ \sin t \end{pmatrix}}$

$f_1(\bar{x}_4(\cdot)) = \int_0^1 \sin t\,dt = 0{,}4597 \qquad f_2(\bar{x}_4(\cdot)) = \int_0^1 (1 + \cos^2 t)^{\frac{1}{2}}\,dt = 1{,}3114$

$\boxed{\bar{x}_5(t) = \begin{pmatrix} \cos \pi t \\ \sin \pi t \end{pmatrix}}$

$f_1(\bar{x}_5(\cdot)) = \int_0^1 -\pi \sin^2 \pi t\,dt = -\frac{\pi}{2} \qquad f_2(\bar{x}_5(\cdot)) = \int_0^1 \pi\,dt = \pi$

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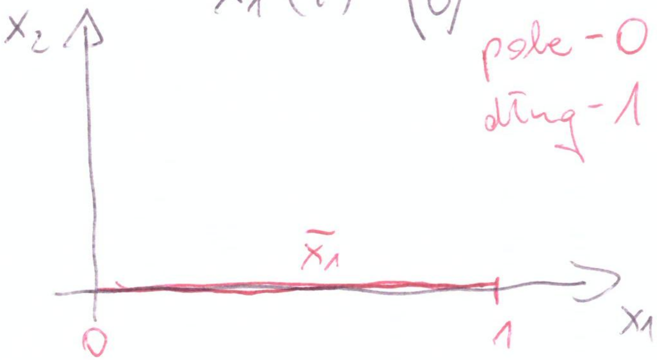
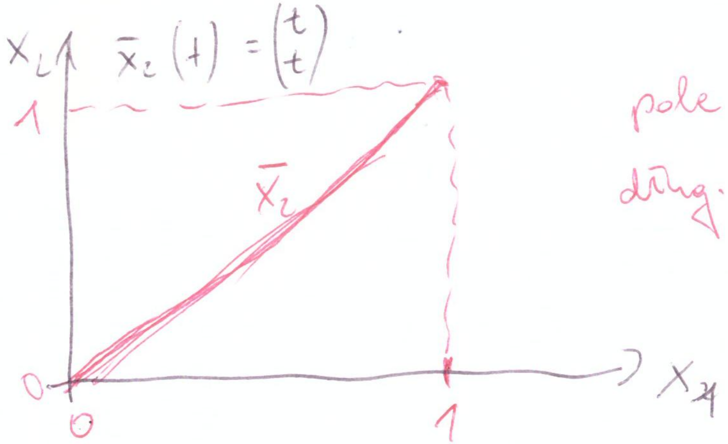
$\overline{x_1}(t) = \begin{pmatrix} t \\ 0 \end{pmatrix}$



pole – 0

dĺž – 1

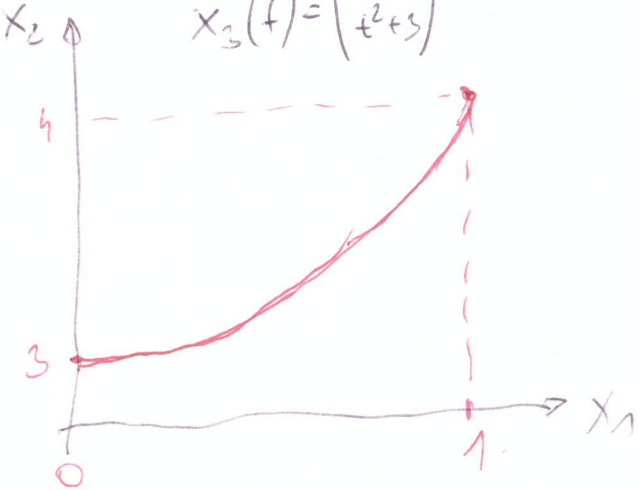
$\overline{x}_2(t) = \begin{pmatrix} t \\ t \end{pmatrix}$



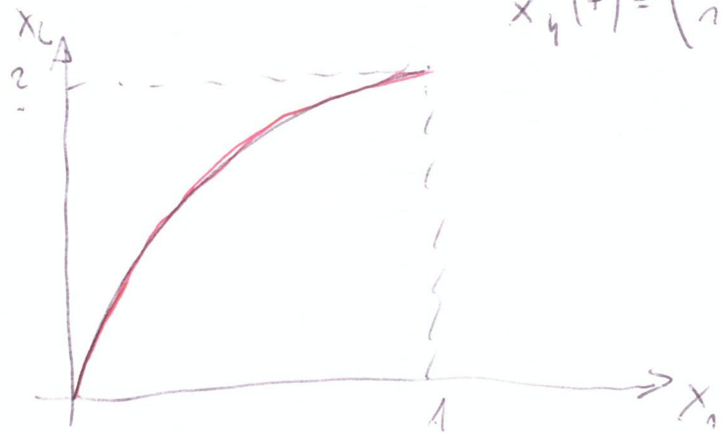
pole – $\frac{1}{2}$

dĺž. – $\sqrt{2}$

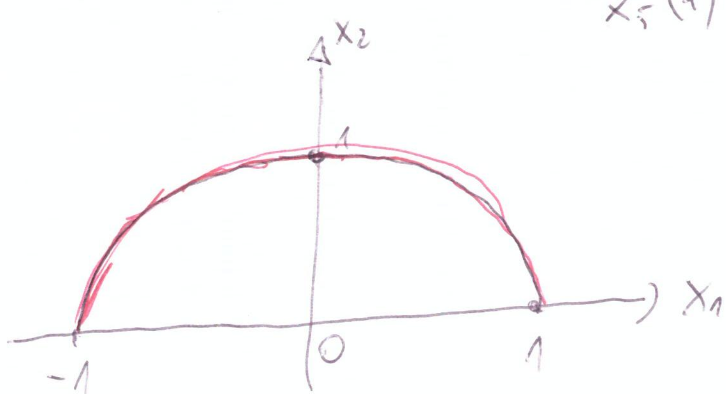
$\overline{x}_3(t) = \begin{pmatrix} t \\ t^2+3 \end{pmatrix}$



$\overline{x}_4(t) = \begin{pmatrix} t \\ \sin t \end{pmatrix}$



$\overline{x}_5(t) = \begin{pmatrix} \cos \pi t \\ \sin \pi t \end{pmatrix}$



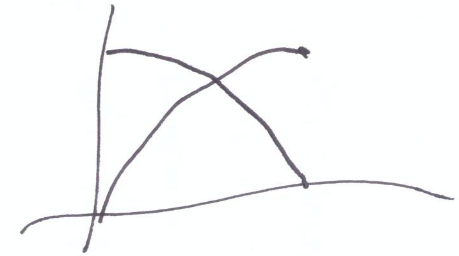
pole – $\frac{\pi}{2}$

dĺž – π

Norma

$\sin(\cdot)$ $\quad [0, \frac{\pi}{2}]$

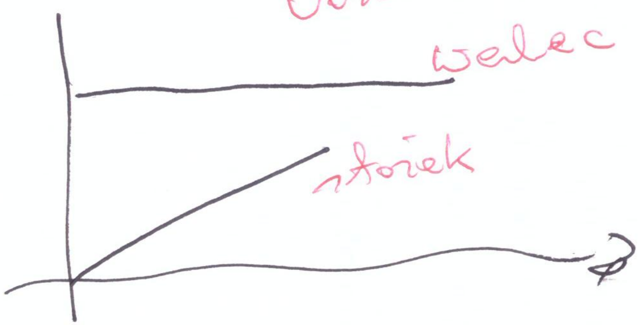
$$\|\sin(\cdot)\|_2 = 1 + 1 + 1 = \underline{3}$$

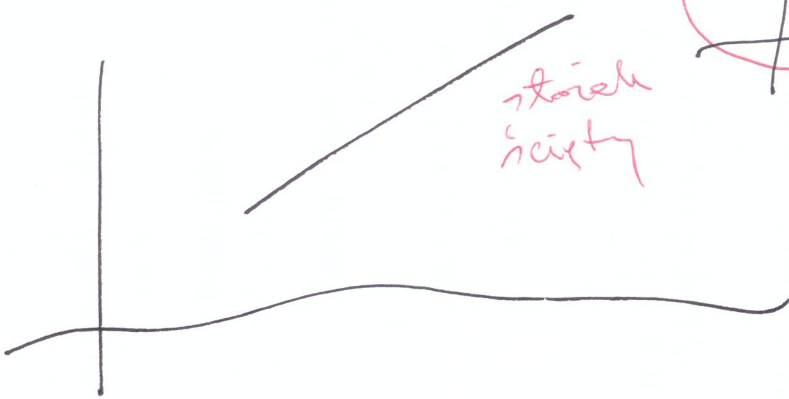


$$\sqrt{\dot{x}_1^2 + \dot{x}_2^2}$$

$x_1(t)$

Obrotowe



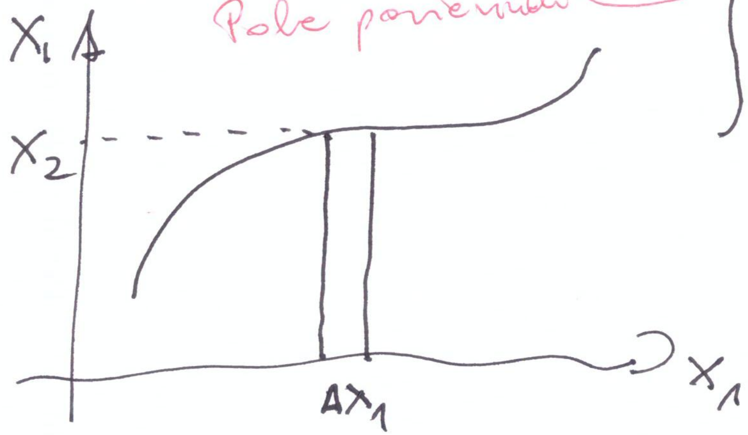


Długość

$$dx_1 = \dot{x}_1 \, dt$$

$$\frac{dx_1}{dt} = \dot{x}_1$$

Pole powierzchni



$$\int x_2 \, dx_1 \cancel{dt} =$$

$$\int_{t_0}^{t_1} x_2 \dot{x}_1 \, dt$$

E1

Dla funkcjonału $f(x(\cdot)) = \boxed{\int_{t_0}^{t_1} L(t, x(t), \dot{x}(t))\,dt}$

pochodna

$$Df(x(\cdot))v(\cdot) = \frac{d}{d\alpha}\Big|_{\alpha=0} \int_{t_0}^{t_1} L(t, \underline{x(t)+\alpha v(t)}, \underline{\dot{x}(t)+\alpha\dot{v}(t)})\,dt =$$

$$= \int_{t_0}^{t_1} \frac{d}{d\alpha}\Big|_{\alpha=0} L(t, x+\alpha v, \dot{x}+\alpha\dot{v})\,dt =$$

$$= \int_{t_0}^{t_1} \left(\frac{\partial L(t, x+\alpha v, \dot{x}+\alpha\dot{v})}{\partial x} v + \frac{\partial L(t, x+\alpha v, \dot{x}+\alpha\dot{v})}{\partial \dot{x}} \dot{v} \right)\Big|_{\alpha=0} dt =$$

$$= \int_{t_0}^{t_1} \left(\frac{\partial L(t, x, \dot{x})}{\partial x} v + \underbrace{\frac{\partial L(t, x, \dot{x})}{\partial \dot{x}} \dot{v}}_{} \right) dt$$

przez części

$\boxed{\int u\dot{v}\,dt = uv - \int \dot{u}v\,dt}$

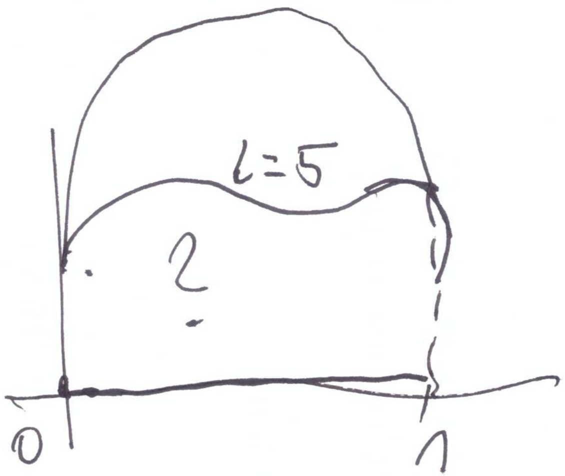
$$\int_{t_0}^{t_1} \underbrace{\frac{\partial L}{\partial \dot{x}}}_{u} \underbrace{\dot{v}}_{\dot{v}}\,dt = \underbrace{\frac{\partial L}{\partial \dot{x}} v \Big|_{t_0}^{t_1}}_{} - \int_{t_0}^{t_1} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) v\,dt$$

$$\downarrow \boxed{\frac{\partial L}{\partial \dot{x}}(t_1)v(t_1) - \frac{\partial L}{\partial \dot{x}}(t_0)v(t_0)} \leftarrow$$

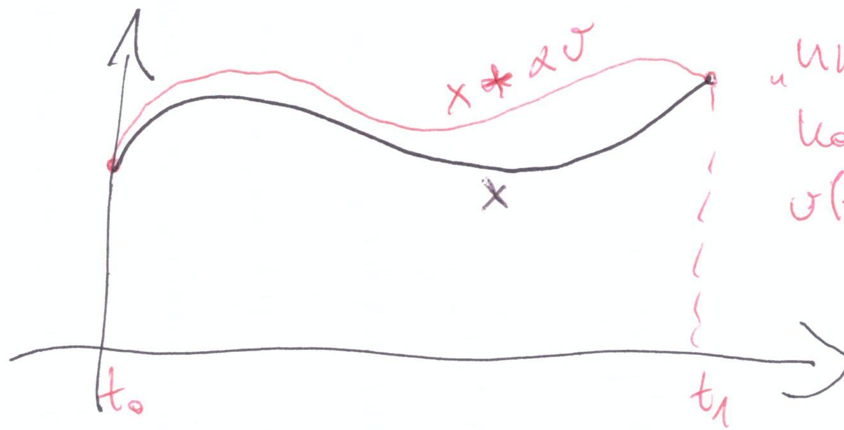
nie chcę tego :(

założam więc $v(t_0) = v(t_1) = 0$ ← „zaburzenie" v (wariacja) krzywej jest zerowe na końcach

$$\boxed{D(f(x(\cdot))v(\cdot) = \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \right) v(t)\,dt}$$ Pochodna



swobodne
końce



„uwięzione"
końce
$v(t_0) = v(t_1) = 0$

E2

$x(\cdot)$ – krzywa

$x(\cdot)+\alpha v(\cdot)$ – jej wariacja

$f(x(\cdot)) = \int_{t_0}^{t_1} L(t, x(t), \dot{x}(t))\, dt$ – funkcjonał

$D(f(x(\cdot))v(\cdot) = \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \right) v(t)\, dt$ – jego pochodna

Warunek na ekstremum

Zasada najmniejszego działania → $\boxed{D(f(x(\cdot))v(\cdot) = 0 \quad \forall v(\cdot)}$

$x(\cdot)$ spełniające – ekstremala funkcjonału

Kiedy $\int_{t_0}^{t_1} coś\, v(t)\, dt = 0$? $\forall v$

$\boxed{coś = 0}$

więc

$$\boxed{\frac{d}{dt}\frac{\partial L(t, x(t), \dot{x}(t))}{\partial \dot{x}} - \frac{\partial L(t, x(t), \dot{x}(t))}{\partial x} = 0}$$

↑ równanie Eulera-Lagrange'a

By znaleźć ekstremum musimy rozwiązać r. r.

Przykład 3.5.2

Najkrótsza krzywa

$$f(x(\cdot)) = \int_{t_0}^{t_1} \underbrace{\boxed{\sqrt{\dot{x}_1^2 + \dot{x}_2^2}}}_{L} \, dt \qquad \boxed{\underbrace{\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x}}_{0} = 0}$$

$$\frac{\partial L}{\partial \dot{x}_1} = \frac{1}{\cancel{2}\sqrt{\dot{x}_1 + \dot{x}_2}} \cancel{2}\dot{x}_1 = \frac{\dot{x}_1}{L} \qquad \frac{\partial L}{\partial x_1} = 0$$

$$\frac{\partial L}{\partial \dot{x}_2} = \frac{\dot{x}_2}{L} \qquad \frac{\partial L}{\partial x_2} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0 \implies \frac{\partial L}{\partial \dot{x}} = C$$

$$\begin{cases} \dfrac{\dot{x}_1}{L} = C_1 & (1) \\ \dfrac{\dot{x}_2}{L} = C_2 & (2) \end{cases} \qquad \frac{(2)}{(1)} = \frac{\dot{x}_2}{\dot{x}_1} = \frac{C_2}{C_1} = C_3$$

$$\frac{\frac{dx_2}{\cancel{dt}}}{\frac{dx_1}{dt}} = \frac{dx_2}{dx_1} = C_3 \qquad \boxed{x_2 = C_3 x_1 + D}$$

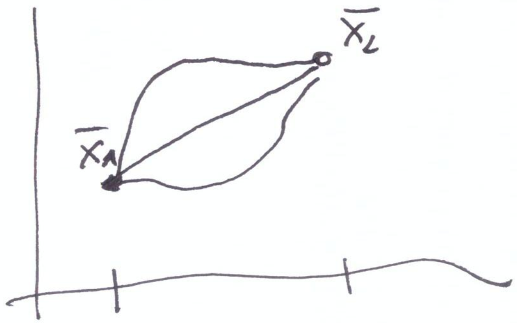
Pole powierzchni bocznej Przykład 3.53

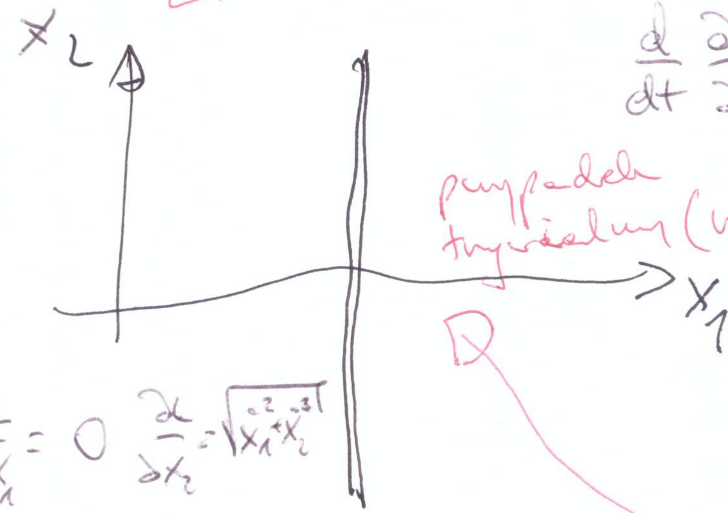
$$f(x(\cdot)) = \int_{t_0}^{t_1} \underbrace{x_2 \sqrt{\dot{x}_1^2 + \dot{x}_2^2}}_{L}\, dt$$

$$L = x_2 \sqrt{\dot{x}_1^2 + \dot{x}_2^2}$$

r EL

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0$$





przypadek trywialny (minimum)

$\Downarrow$ dla x_1 $\frac{\partial L}{\partial x_1} = 0$

więc

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} = 0 \Rightarrow \frac{\partial L}{\partial \dot{x}_1} = C$$

$$\frac{\partial L}{\partial \dot{x}_1} = \frac{x_2 \dot{x}_1}{\sqrt{\dot{x}_1 + \dot{x}_2}} \quad \frac{\partial L}{\partial \dot{x}_2} = \frac{x_2 \dot{x}_2}{\sqrt{\dot{x}_1 + \dot{x}_2}} \quad \frac{\partial L}{\partial x_1} = 0 \quad \frac{\partial L}{\partial x_2} = \sqrt{\dot{x}_1^2 + \dot{x}_2^2}$$

równanie

$$\begin{cases} \frac{x_2 \dot{x}_1}{\sqrt{\dot{x}_1^2 + \dot{x}_2^2}} = C \to \frac{x_2}{\sqrt{\dot{x}_1^2 + \dot{x}_2^2}} = \frac{C}{\dot{x}_1} \\ \frac{d}{dt}\left(\frac{x_2 \dot{x}_2}{\sqrt{\dot{x}_1^2 + \dot{x}_2^2}}\right) - \sqrt{\dot{x}_1^2 + \dot{x}_2^2} = 0 \end{cases}$$

$$\frac{d}{dt}\left(\frac{C\dot{x}_2}{\dot{x}_1}\right) = \sqrt{\dot{x}_1^2 + \dot{x}_2^2} = \frac{x_2 \dot{x}_1}{C} \quad |:C \quad C \to \frac{1}{\alpha}$$

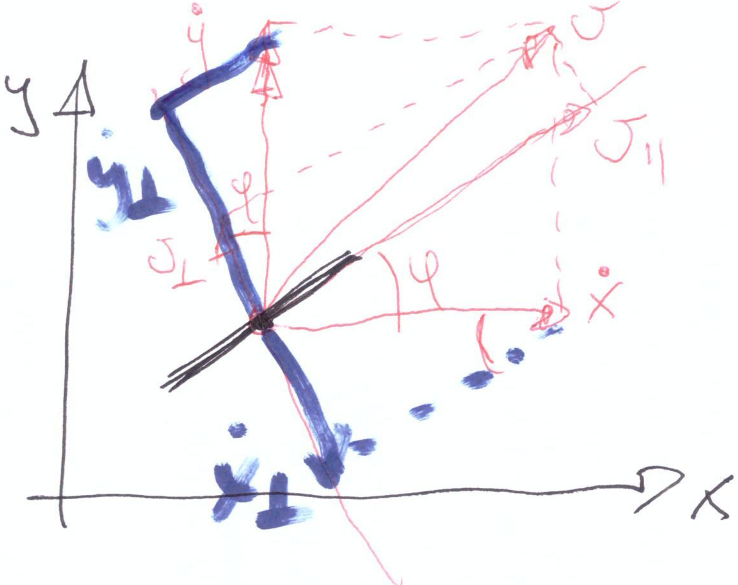
r.r 2 rzędu

liniowy równanie charakter. wystarczy i mamy

$$\frac{d}{dt}\frac{dx_2}{dx_1} = \frac{d^2 x_2}{dx_1^2}\dot{x}_1 = \alpha^2 x_2 \dot{x}_1 \Rightarrow \left(\frac{d^2 x_2}{dx_1^2} - \alpha x_2\right)\dot{x}_1 = 0$$

2 przypadki: albo $\boxed{\frac{d^2 x_2}{dx_1^2} - \alpha x_2 = 0}$ albo $\dot{x}_1 = 0 \to x_1 = C_1$ ← linie pionowe pow boczne $= 0$

$x_2 = c \cosh(\alpha x_1 + \beta)$



$$q = \begin{pmatrix} x \\ y \\ \varphi \end{pmatrix}$$

brak poślizgu poprzecznego

wer brak pośl. bocz

$\upsilon_{\perp} = 0$

$$\upsilon_{\perp} = \dot{x}_{\perp} - \dot{y}_{\perp}$$

$$\dot{x}\sin\varphi - \dot{y}\cos\varphi = 0$$

$$f(q, \dot{q}) = 0$$