

SCIENTIFIC & ENGINEERING PROGRAMMING

II Year Electronics and Computer Engineering, FoEPhaM, WUST

6 Mathematica Lab Class 6 – Dynamical systems

6.1 The scope

To get familiar with the methodology of dynamical systems simulations in Mathematica, methods for results visualization and analysis.

6.2 Prerequisites

Before the classes you should know, how to:

- model simple physical systems,
- represent and define differential equations,
- solve numerically differential equations,
- visualize functions being the differential equations solutions,
- visualize the results on animations.

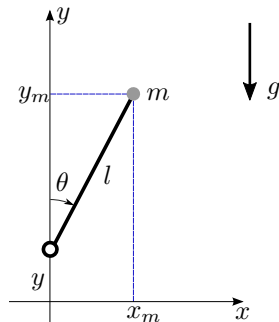
Read also carefully the tasks preliminaries. Analyze the “To dos”. Decide which task is more suitable for you to be performed.

6.3 Tasks

6.3.1 Inverted Pendulum with Moving Vertically Base

Preliminaries

Differential equations, describing the dynamical behavior of a depicted below inverted pendulum of the length l and the mass m , with a moving vertically base,



has a form

$$l\ddot{\theta} = g\sin\theta - c\dot{\theta} + \sin\theta\ddot{y}, \quad (1)$$

where θ denotes the angle the pendulum link makes with the vertical, y is the base offset, c viscous damping coefficient, and g stands for the gravitational acceleration. The dot operator means the time derivative, e.g. $\dot{\theta} = \frac{d\theta}{dt}$.

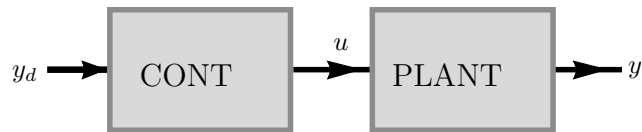


Figure 1: Open loop control scheme

Let's assume, the pendulum base is moving in an oscillatory way, described by the equation*

$$y(t) = A \sin(\omega t), \quad (2)$$

where A is the oscillations amplitude and ω their frequency. Introduction of the oscillatory base (2) into the system (1) allows to control its behavior exploiting the idea of an open loop (feedforward) control scheme, illustrated in Figure 1. In the figure PLANT stands for the object to be controlled, y is its output, and u its input. CONT is a controller, which according to the desired system behavior (i.e. its desired output value y_d – see Figure 1) tries to generate the control input u causing the system to behave as desired (i.e. causing the output y to approach y_d .) In the case of the analyzed inverted pendulum with a moving vertically base, the role of the input u plays the second derivative of the base position \ddot{y} , and the angle θ is taken as the output y . The oscillator takes the role of the controller. Changing its oscillations frequency and amplitude one can control the behavior of the pendulum and decide, which equilibrium the pendulum stabilizes at (oscillates around).

To do

The task is to simulate and visualize the system (1) and to perform some experimental simulations with controlling it. For the described system do the following:

1. Prepare the Mathematica model of the inverted pendulum with moving vertically base:
 - (a) define the differential equation (1) modeling the pendulum together with initial conditions,
 - (b) define a function which represents the pendulum graphically.

Analyze the model behavior for the initial value of θ equal to 0^\dagger and 0.2 , and the steady base ($y=0$). Plot the joint angle as a function of time. Animate the pendulum movement. Investigate the influence of the dumping coefficient on the pendulum behavior.

2. Equip the pendulum with the oscillator (2):
 - (a) define the rule which introduces the controller (2),
 - (b) apply the defined rule to the system.

Examine the correctness of the provided solution.

3. Simulate the system behavior for both the initial values from Point 1 with different values of oscillations amplitude and frequency. Try for example[‡] $A = 0.2, \omega = 4$; $A = 0.2, \omega = 40$; $A = 0.2, \omega = 400$; $A = 0.02, \omega = 400$; $A = 0.002, \omega = 400$; $A = 0.002, \omega = 4000$. Try to derive some conclusions about the influence of the oscillator amplitude and its frequency on the system behavior[§].

*An inverted pendulum with such a base is known as Kapitza's pendulum.

[†]For the practical purposes you can observe also the pendulum behavior for the initial value very close to 0 (e.g. 10^{-4} .)

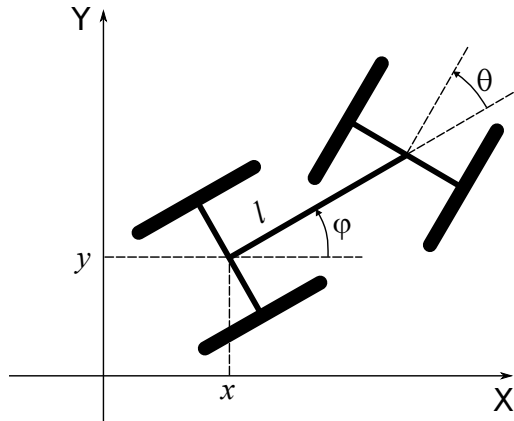
[‡]With the pendulum length $l = 1$ and the dumping coefficient $c = 0.2$.

[§]Do not forget to change the animation speed if necessary, to observe the actual system behavior

6.3.2 Kinematic car

Preliminaries

Consider the kinematic car, as shown below,



where $(x, y, \varphi, \theta)^T$ is the depending on time t car configuration (x, y being the rear axle position, φ – the car orientation, and θ – the wheel steering angle,) and l is its parameter (length). The kinematics model of such car is given by a set of differential equations

$$\begin{cases} \dot{x} = \cos \theta \cos \varphi u_1 \\ \dot{y} = \cos \theta \sin \varphi u_1 \\ \dot{\varphi} = \frac{\sin \theta}{l} u_1 \\ \dot{\theta} = u_2 \end{cases}, \quad (3)$$

where u_1 and u_2 are the car controls (its longitudinal speed, and the front axle turning speed), and a dot denotes the time derivative, e.g. $\dot{x} = \frac{dx}{dt}$.

With the control input u one can control the car behavior exploiting the idea of an open loop (feedforward) control scheme, illustrated in Figure 1 and described on page 2.

To do

The task is to simulate and visualize the system (3) and to perform some experimental simulations with controlling it. For the described system do the following:

1. Prepare the Mathematica model of the kinematic car :
 - (a) define the differential equation (3) modeling the car together with initial conditions,
 - (b) define a function which represents the car graphically[¶].

Analyze the model behavior with $l = 1$ for different initial conditions and constant controls. Prepare the simulations environment allowing for results visualization (time plots, animation.) Interpret the obtained results.

2. Investigate the influence of the car length l on its behavior.
3. Observe the behavior of the car for a sequences of constant controls as follows (apply each value for a constant periods t_0 (start with $t_0 = 1$, then try values from the range $[0.1s, 1s]$)[¶]):

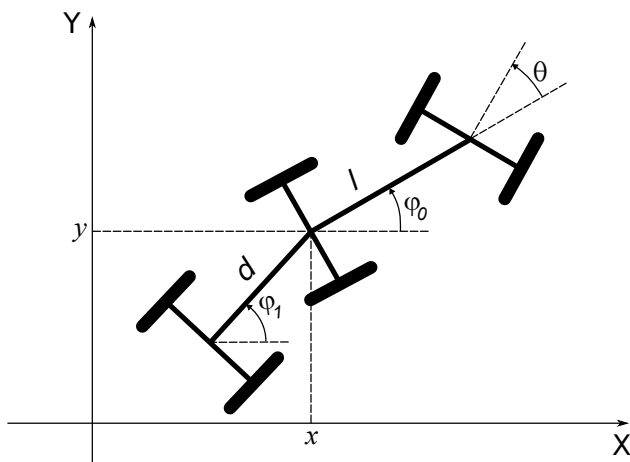
[¶]Before the classes describe at least the car geometry – the car can be represented by 3 straight line segments being a “car body” of length l , and its two axles: front (turnable) and rear (in the simplest case both without “wheels” but which can be added as 4 additional line segments). Compute the axles’ end coordinates as a function of the car configuration vector $(x, y, \varphi, \theta)^T$.

[¶]Use the function provided on the course web page in the laboratory classes table.

- (a) $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix}$
- (b) $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
- (c) $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix}$
- (d) $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$
- (e) $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ -2 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

4. Simulate the behavior of the car for periodic controls in the form $\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} a_1 \sin(\omega_1 t + \omega_{10}) \\ a_2 \sin(\omega_2 t + \omega_{20}) \end{pmatrix}$, for some constant parameters $a_i, \omega_i, \omega_{i0}, i = 1, 2$.

5. Equipping a kinematic car with a trailer, as shown below,



where $(x, y, \varphi_0, \varphi_1, \theta)$ is the system configuration (x, y being the rear axle position, φ_0 – the car orientation, φ_1 – the trailer orientation, and θ – the wheel steering angle,) and l and d are its parameters (car and trailer lengths, respectively,) one obtains the kinematics model of such system described by a set of differential equations

$$\begin{cases} \dot{x} = \cos \theta \cos \varphi_0 u_1 \\ \dot{y} = \cos \theta \sin \varphi_0 u_1 \\ \dot{\varphi}_0 = \frac{\sin \theta}{l} u_1 \\ \dot{\varphi}_1 = \frac{\cos \theta}{d} \sin(\varphi_0 - \varphi_1) u_1 \\ \dot{\theta} = u_2 \end{cases},$$

where u_1 and u_2 are the car controls (its longitudinal speed, and the front axle turning speed.) Repeat for this system the simulations from the tasks 1–4.