Scientific & Engineering Programming

II Year Electronics and Computer Engineering, FoEPhaM, WUST

4 Mathematica Lab Class 4 – ODEs

4.1 The scope

To get familiar with ordinary differential equations in Mathematica, methods for solving them, to perform basic results visualization and analysis.

4.2 Prerequisites

Before the classes you should know, how to:

- differentiate and integrate expressions,
- plot expressions/functions
- represent and define differential equations,
- solve symbolically and numerically differential equations,
- visualize functions being the differential equations solutions,
- model simple physical systems.

To help understanding physical systems modeling watch "Chaos: The Science of the Butterfly Effect" movie (link in Materials column, at least starting form 2:10, paying particular attention to the concept of a phase space/portrait (2:40)).

4.3 Tasks

4.3.1 Functions differentiation/integration

1. Define the functions $f(x) = \frac{x}{x^4+1}$, $g(x, y) = 25 - x^2 - y^3$, and $h(x, y) = \begin{pmatrix} x + 2y \\ xy \end{pmatrix}$.

   (a) Calculate their derivatives: $\frac{df(x)}{dx}$, $\frac{\partial g(x,y)}{\partial x}$, $\frac{\partial g(x,y)}{\partial y}$, $\frac{\partial h(x,y)}{\partial x}$, $\frac{\partial h(x,y)}{\partial y}$.

   (b) Calculate their integrals: $\int f(x)dx$, $\int_{-3}^{3} f(x)dx$, $\int g(x, y)dx$, $\int_{-3}^{3} g(x, y)dx$, $\int g(x, y)dy$, $\int_{-3}^{3} g(x, y)dy$, $\int h(x, y)dx$, $\int_{-3}^{3} h(x, y)dx$, $\int h(x, y)dy$, $\int_{-3}^{3} h(x, y)dy$.

   (c) Plot the above on the interval $[-3, 3]$.

2. For functions $f(x), g(x) \in \mathbb{C}([a, b])$ (i.e. $f, g : [a, b] \to \mathbb{R}$ and they are continuous on $[a, b]$) we define the inner product as

$$\langle f(x), g(x)\rangle = \int_a^b f(x)g(x)dx.$$

   Two functions f and g are orthogonal if and only if $\langle f(x), g(x)\rangle = 0$.

   (a) Verify (symbolically/numerically) that the functions 1, $\cos(x)$, $\sin(x)$ are orthonormal on $[0, 2\pi]$. If not, normalise them.

(b) Verify (symbolically/numerically) that the functions $1, \cos(x), \sin(x), \ldots, \cos(nx), \sin(nx)$, $n \in \mathbb{N}$ are orthonormal on $[0, 2\pi]$. If not, normalise them.

3. Compute derivatives/integrals of functions from the tasks 7-10, Lab Class 3 if possible. Plot them.

4.3.2 Differential equations

4. Solve symbolically (with `DSolve` function) the second order differential equation

$$y''(x) + py'(x) + qy(x) = 0$$

with initial conditions set to $y(0) = y_0$, $y'(0) = yp_0$, and:

(a) parameters $p = 3$, $q = 1$. Plot the result $y(x)$ for initial conditions $y_0 = 5$, $yp_0 = 0$ with $x \in [0, 10]$. Plot the result for initial conditions $y_0 = -5, -4, \ldots, 5$, $yp_0 = 0$ (on a single chart). Plot the result for initial conditions $y_0 = 5$, $yp_0 = -5, -4, \ldots, 5$ (on a single chart). Interpret the results.

(b) parameters $p = 1$, $q = 3$. Plot the same graphs as in point 4a. Interpret the results. How have the system behaviour changed? What is the system behaviour for parameters $p = 0$, $q = 3$?

(c) parameters p and q with no assigned numerical values (fully symbolic case). Plot the result for parameters $p = 3$, $q = 1$ and initial conditions $y_0 = 5$, $yp_0 = 0$. Plot the result for parameters $p = 0.5, 1.5, \ldots, 5.5$, $q = 1$ and initial conditions $y_0 = 5$, $yp_0 = 0$. Analyse the system behaviour for different values of p and q, such that the characteristic equation discriminant ($D = p^2 - 4q$) is larger than, smaller than, equal to 0. Visualize the obtained results. What is the system behaviour for parameters $p = 0$, $q = 1$?

5. Repeat the computation of task 4 utilizing the numerical methods – `NDSolve` function (observe, in this case one has to assign the numerical values to initial conditions y_0, yp_0 first and then solve the equations). Visualize the results and compare with the results from the previous exercise.

6. Solve numerically (and optionally symbolically) the set of equations

$$\begin{cases} \frac{dx_1}{dt} = x_2 \\ \frac{dx_2}{dt} = -x_1 - kx_2 \end{cases}.$$

Visualize the result with time plot and phase portrait (state space plot) and interpret the results for different initial conditions and values of the parameter k (large: $k \geq 2$ and small: $0 \leq k < 2$).

7. Solve numerically the set of equations

$$\begin{cases} \frac{dx}{dt} = \sigma(y - x) \\ \frac{dy}{dt} = x(\rho - z) - y \\ \frac{dz}{dt} = xy - \beta z \end{cases}.$$

Visualize the results (time plot, state space plot, xz plane) for different system parameters (start with $\sigma = 10$, $\beta = 8/3$, $\rho = 28$). Examine the influence of initial conditions to the system trajectory (one may start with $(0, 1, 0)$, what Lorenz considered :).

8. Using the Newton's laws write the differential equation of a harmonic oscillator of the mass m, which is a system that, when displaced from its equilibrium position, experiences a restoring force F, proportional to the displacement x:

$$F = -kx,$$

where k is a positive constant. Solve the equation and visualizing the obtained solution determine the motion characteristics.