Mechanika analityczna, II r AiR inż.

Wykład 5

Mechanika Lagranżowska

Układ opisujemy współrzędnymi $q \in \mathbb{R}^n$ i prędkościami uogólnionymi $\dot{q} \in \mathbb{R}^n$. Definiujemy funkcję zwaną lagranżianem

$$L(t,q,\dot{q}) = K(t,q,\dot{q}) - V(q)$$

będącą różnicą energii kinetycznej i energii potencjalnej.

Ruch układu podlega zasadzie wariacyjnej – Zasadzie Najmniejszego Działania, która orzeka, że trajektoria układu jest punktem stacjonarnym (minimum) działania

$$I(q(\cdot)) = \int_{t_0}^{t_1} L(t,q,\dot{q})\,dt.$$

Zasada Najmniejszego Działania pochodzi od Hamiltona.

Równania ruchu układu otrzymuje się z równań Eulera-Lagrange'a dla lagranżianu:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0 \; (F) \quad F\text{-siły niepotencjalne}$$

Równania ruchu są 2-rzędu

$$\frac{\partial^2 L}{\partial \dot{q}^2}\ddot{q} + \frac{\partial^2 L}{\partial q \partial \dot{q}}\dot{q} - \frac{\partial L}{\partial q} = 0.$$

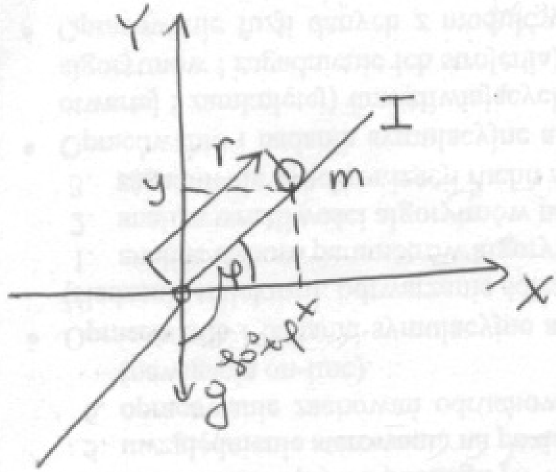
Jeśli macierz Hessego $\frac{\partial^2 L}{\partial \dot{q}^2}$ jest odwracalna, to

$$\ddot{q} = +\left(\frac{\partial^2 L}{\partial \dot{q}^2}\right)^{-1}\left(\frac{\partial L}{\partial q} - \frac{\partial^2 L}{\partial q \partial \dot{q}}\dot{q}\right) \Longrightarrow q(t).$$

Przykłady

Ex 1 Belka i kule

$$V = -m(\vec r, \vec g) = -mrg\cos\varphi = mrg\sin\varphi$$



$$q = (r, \varphi), \quad \dot q = (\dot r, \dot\varphi)$$

$$K = \frac{1}{2}mv^2 + \frac{1}{2}I\dot\varphi^2, \quad V = +mgy = +mgr\sin\varphi$$

$$v^2 = \dot x^2 + \dot y^2$$

$$x = r\cos\varphi \qquad \dot x = \dot r\cos\varphi - r\dot\varphi\sin\varphi$$

$$y = r\sin\varphi \qquad \dot y = \dot r\sin\varphi + r\dot\varphi\cos\varphi$$

$$v^2 = \dot r^2 + r^2\dot\varphi^2$$

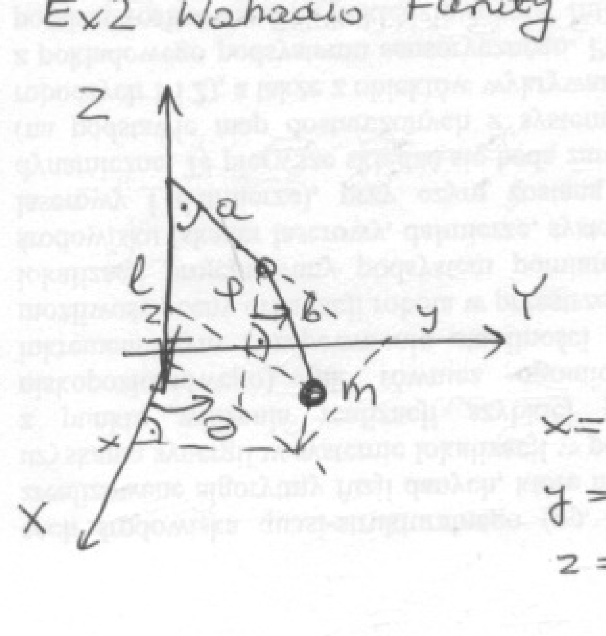
$$L = K - V = \frac{1}{2}(mr^2 + I)\dot\varphi^2 + \frac{1}{2}m\dot r^2 + mgr\sin\varphi$$

$$\frac{\partial L}{\partial r} = mr\dot\varphi^2 + mg\sin\varphi \qquad \frac{\partial L}{\partial \dot r} = m\dot r$$

$$\frac{\partial L}{\partial \varphi} = -mgr\cos\varphi \qquad \frac{\partial L}{\partial \dot\varphi} = (mr^2 + I)\dot\varphi$$

$$\begin{cases} \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot r} - \dfrac{\partial L}{\partial r} = m\ddot r - mr\dot\varphi^2 + mg\sin\varphi = 0 \; (F_1) \\ \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot\varphi} - \dfrac{\partial L}{\partial \varphi} = (mr^2 + I)\ddot\varphi + 2mr\dot r\dot\varphi + mgr\cos\varphi = 0 \; (M_2) \end{cases}$$

Ex 2 Wahadło Furuty



$$q = (\theta, \varphi), \quad \dot q = (\dot\theta, \dot\varphi)$$

$$K = \frac{1}{2}mv^2 \qquad V = +mgz$$

$$v^2 = \dot x^2 + \dot y^2 + \dot z^2$$

$$x = (a + b\sin\varphi)\cos\theta$$

$$y = (a + b\sin\varphi)\sin\theta$$

$$z = l - b\cos\varphi$$

$$\dot{x} = -(a+b\sin\varphi)\sin\theta\,\dot{\theta} + b\cos\varphi\cos\theta\,\dot{\varphi}$$

$$\dot{y} = (a+b\sin\varphi)\cos\theta\,\dot{\theta} + b\cos\varphi\sin\theta\,\dot{\varphi}$$

$$\dot{z} = b\sin\varphi\,\dot{\varphi}$$

$$v^2 = (a+b\sin\varphi)^2\dot{\theta}^2 + b^2\dot{\varphi}^2$$

$$L = \tfrac{1}{2}m(a+b\sin\varphi)^2\dot{\theta}^2 + \tfrac{1}{2}mb^2\dot{\varphi}^2 + mg(l-b\cos\varphi)$$

$$\frac{\partial L}{\partial \theta} = 0 \qquad \frac{\partial L}{\partial \dot{\theta}} = m(a+b\sin\varphi)^2\dot{\theta}$$

$$\frac{\partial L}{\partial \varphi} = \cancel{2}m(a+b\sin\varphi)b\cos\varphi\,\dot{\theta}^2 + mgb\sin\varphi \qquad \frac{\partial L}{\partial \dot{\varphi}} = mb^2\dot{\varphi}$$

$$\begin{cases} \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{\theta}} = 0 \Rightarrow m(a+b\sin\varphi)^2\dot{\theta} = const \quad ZZMP \\ \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{\varphi}} - \dfrac{\partial L}{\partial \varphi} = mb^2\ddot{\varphi} - \cancel{2}m(a+b\sin\varphi)b\cos\varphi\,\dot{\theta}^2 + mgb\sin\varphi = 0 \quad (M_z) \end{cases}$$

W ten sposób uzyskaliśmy równania ruchu układu

Ogólna postać równań ruchu

Dla typowych układów $K = \frac{1}{2}\sum_{ij} Q_{ij}\dot{q}_i\dot{q}_j = \frac{1}{2}\dot{q}^T Q(q)\dot{q}$

– energia kinetyczna jest formą kwadratową prędkości, której macierz zależy od położenia. Zatem

$$L(t,q,\dot{q}) = \frac{1}{2}\sum_{ij} Q_{ij}(q)\dot{q}_i\dot{q}_j - V(q)$$

Liczymy $\dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}_k} - \dfrac{\partial L}{\partial q_k}$ dla $k = 1,2,\dots,n$

$$\frac{\partial L}{\partial \dot{q}_k} = \frac{1}{2}\sum_{ij} Q_{ij}\dot{q}_i\delta_{jk} + \frac{1}{2}\sum_{ij} Q_{ij}\dot{q}_j\delta_{ik} = \frac{1}{2}\sum_i Q_{ik}\dot{q}_i + \frac{1}{2}\sum_j Q_{kj}\dot{q}_j = \sum_i Q_{ik}(q)\dot{q}_i$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = \sum_i Q_{ik}(q)\ddot{q}_i + \frac{1}{2}\sum_{ij}\frac{\partial Q_{ik}}{\partial q_j}\dot{q}_i\dot{q}_j + \frac{1}{2}\sum_{ij}\frac{\partial Q_{kj}}{\partial q_i}\dot{q}_i\dot{q}_j$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q_k} = \sum_i Q_{ki}(q)\ddot q_i + \sum_{ij} \frac{1}{2}\left(\frac{\partial Q_{ik}}{\partial q_j} + \frac{\partial Q_{kj}}{\partial q_i}\right)\dot q_i \dot q_j$$

$$\frac{\partial L}{\partial q_k} = \sum_{ij} \frac{\partial Q_{ij}}{\partial q_k}\dot q_i \dot q_j - \frac{\partial V}{\partial q_k}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q_k} = \sum_i Q_{ki}\ddot q_i + \sum_{ij} c_{ij}^k(q)\dot q_i \dot q_j + \frac{\partial V}{\partial q_k} = O(F_k),$$

gdzie $c_{ij}^k(q) = \frac{1}{2}\left(\frac{\partial Q_{ik}}{\partial q_j} + \frac{\partial Q_{kj}}{\partial q_i} - \frac{\partial Q_{ij}}{\partial q_k}\right)$ – symbole Christoffela I rodzaju.

W postaci macierzowej

$$Q(q)\ddot q + C(q,\dot q)\dot q + D(q) = O(F), \quad (*)$$

gdzie $Q(q) = [Q_{ij}(q)]$ – macierz inercji układu

$C(q,\dot q)$ – macierz sił Coriolisa i odśrodkowych

$D(q)$ – wektor sił bezwładności

F – wektor sił niepotencjalnych (sterowań)

Mamy $(C(q,\dot q)\dot q)_k = \sum_j C_{kj}(q,\dot q)\dot q_j = \sum_{ij} c_{ij}^k(q)\dot q_i \dot q_j$.

Stąd

$$\boxed{C(q,\dot q)_{kj} = \sum_i c_{ij}^k(q)\dot q_i}$$

Równania ruchu (*) można zapisać w postaci

$$\ddot q_k + \sum_{ij}\Gamma_{ij}^k(q)\dot q_i \dot q_j + \bar D_k(q) = \bar F_k,$$

gdzie $\Gamma_{ij}^k(q)$ – symbole Christoffela II rodzaju

$$(Q^{-1}(q)C(q,\dot q)\dot q)_k = \sum_l Q^{-1}_{kl}(C(q,\dot q)\dot q)_l = \sum_{lij} Q^{-1}_{kl} c_{ij}^l \dot q_i \dot q_j =$$

$$= \sum_{ij}\Gamma_{ij}^k(q)\dot q_i \dot q_j$$

Stąd

$$\boxed{\Gamma_{ij}^{k}(q) = \sum_{l} Q_{kl}^{-1} C_{ij}^{l}(q)}.$$

Własności: $\quad C_{ij}^{k}(q) = C_{ji}^{k}(q)$

$$\dot{Q} = C + C^T$$

$$\dot{Q}_{kj} = \sum_{i} \frac{\partial Q_{kj}}{\partial q_i} \dot{q}_i$$

$$C_{kj} = \sum_{i} C_{ij}^{k} \dot{q}_i \qquad C_{jk} = \sum_{i} C_{ik}^{j} \dot{q}_i$$

$$C_{kj} + C_{jk} = \sum_{i} (C_{ij}^{k} + C_{ik}^{j}) \dot{q}_i$$

$$C_{ij}^{k} + C_{ik}^{j} = \frac{1}{2}\left(\frac{\partial Q_{ik}}{\partial q_j} + \cancel{\frac{\partial Q_{kj}}{\partial q_i}} - \cancel{\frac{\partial Q_{ij}}{\partial q_k}}\right) + \frac{1}{2}\left(\cancel{\frac{\partial Q_{ij}}{\partial q_k}} + \frac{\partial Q_{kj}}{\partial q_i} - \cancel{\frac{\partial Q_{ik}}{\partial q_j}}\right)$$

$$= \frac{\partial Q_{kj}}{\partial q_i}$$

Jeżeli $L = K = \frac{1}{2} \sum_{ij} Q_{ij}(q) \dot{q}_i \dot{q}_j$, to wzdłuż trajektorii

$$Q\ddot{q} + C(q, \dot{q})\dot{q} = 0$$

$\frac{dL}{dt} = 0$. Albowiem

$$\frac{dL}{dt} = \frac{d}{dt} \frac{1}{2} \dot{q}^T Q(q) \dot{q} = \dot{q}^T Q(q) \ddot{q} + \frac{1}{2} \dot{q}^T \dot{Q} \dot{q} =$$

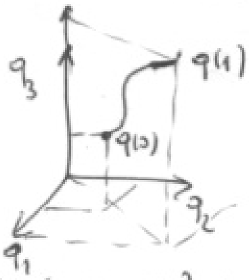
$$= \frac{1}{2} \dot{q}^T \dot{Q} \dot{q} - \dot{q}^T C(q, \dot{q}) \dot{q} = \frac{1}{2} \dot{q}^T (\dot{Q} - 2C) \dot{q} =$$

$$= \frac{1}{2} \dot{q}^T \underbrace{(C^T - C)}_{\text{m. sk. symetryczna.}} \dot{q} = 0$$

Oznacza to, że $\quad L(q, \dot{q}) = \text{const.} = K.$

Przykład: integrator Brocketta – sterowanie optymalne – zadanie nieholonomiczne

$$\begin{cases} \dot q_1 = u_1 \\ \dot q_2 = u_2 \\ \dot q_3 = q_1 u_2 - q_2 u_1 \end{cases}$$



Problem: znaleźć sterowanie $u(t) = (u_1(t), u_2(t))$, takie że układ przechodzi ze stanu $q(0)$ do stanu $q(1) = q_d$ w taki sposób, że

$$\int_0^1 (u_1^2(t) + u_2^2(t))\,dt \longrightarrow \min$$

Przeformułowanie problemu: ponieważ $u_1 = \dot q_1$, $u_2 = \dot q_2$, szukamy krzywej $q_1(t), q_2(t), q_3(t)$, które ekstremalizuje funkcjonał

$$\int_0^1 (\dot q_1^2 + \dot q_2^2)\,dt = \int_0^1 L(q, \dot q)\,dt$$

przy warunku $\quad \dot q_3 - q_1 \dot q_2 + q_2 \dot q_1 = G(q, \dot q) = 0.$

$$\tilde L(q, \dot q, \lambda) = \dot q_1^2 + \dot q_2^2 + \lambda(\dot q_3 - q_1 \dot q_2 + q_2 \dot q_1)$$

$$\frac{\partial \tilde L}{\partial \dot q_1} = 2\dot q_1 + \lambda q_2, \quad \frac{\partial \tilde L}{\partial \dot q_2} = 2\dot q_2 - \lambda q_1, \quad \frac{\partial \tilde L}{\partial \dot q_3} = \lambda$$

$$\frac{\partial \tilde L}{\partial q_1} = -\lambda \dot q_2, \quad \frac{\partial \tilde L}{\partial q_2} = \lambda \dot q_1, \quad \frac{\partial \tilde L}{\partial q_3} = 0$$

Stąd:

$$2\ddot q_1 + \dot\lambda q_2 + \lambda \dot q_2 + \lambda \dot q_2 = 0$$
$$2\ddot q_2 - \dot\lambda q_1 - \lambda \dot q_1 - \lambda \dot q_1 = 0$$
$$\dot\lambda = 0 \quad \Rightarrow \lambda = \text{const}$$

Wobec tego mamy
$$\ddot q_1 + \lambda \dot q_2 = 0 \qquad \dot q_1 = u_1$$
$$\ddot q_2 - \lambda \dot q_1 = 0 \qquad \dot q_2 = u_2$$

$$\left.\begin{aligned} \dot u_1 + \lambda u_2 = 0 &\Rightarrow \ddot u_1 + \lambda \dot u_2 = \ddot u_1 + \lambda^2 u_1 = 0 \\ \dot u_2 - \lambda u_1 = 0 &\Rightarrow \ddot u_2 + \lambda^2 u_2 = 0 \end{aligned}\right\} \text{równanie liniowe, jedn. rzędu 2}$$

$\ddot u_1 + \lambda^2 u_1 = 0 \qquad$ r. ch. $\; s^2 + \lambda^2 = 0 \Rightarrow s = \pm i\lambda$

$$\begin{cases} u_1(t) = C_1 \cos\lambda t + C_2 \sin\lambda t = \dot q_1 \\ u_2(t) = D_1 \cos\lambda t + D_2 \sin\lambda t = \dot q_2 \end{cases} \qquad \cancel{\dot q_1(0) = C_1} \Rightarrow \quad \cancel{\dot q_2(0) = \ldots} =$$

Wyliczamy trajektorie:

$$\dot q_1 = u_1 = C_1\cos\lambda t + C_2\sin\lambda t \qquad \dot q_2 = u_2 = D_1\cos\lambda t + D_2\sin\lambda t$$

$$q_1(t) = \frac{C_1}{\lambda}\sin\lambda t - \frac{C_2}{\lambda}\cos\lambda t \Big|_0^t + q_1(0)$$

$$\underline{q_1(t) = q_1(0) + \frac{C_1}{\lambda}\sin\lambda t - \frac{C_2}{\lambda}\cos\lambda t + \frac{C_2}{\lambda}}$$

$$\underline{q_2(t) = q_2(0) + \frac{D_1}{\lambda}\sin\lambda t - \frac{D_2}{\lambda}\cos\lambda t + \frac{D_2}{\lambda}}$$

Weźmy $q_1(0)=q_2(0)=q_3(0)=0$, $q_1(1)=q_2(1)=0$, $q_3(1)=a\neq 0$

$$q_1(1) = \frac{C_1}{\lambda}\sin\lambda - \frac{C_2}{\lambda}\cos\lambda + \frac{C_2}{\lambda} = 0 \Rightarrow \lambda = k\cdot 2\pi \qquad k = 0, \pm 1, \pm 2, \ldots$$

$q_2(1) = 0$ – to samo.

Mamy
$$q_1(t) = \frac{C_1}{\lambda}\sin\lambda t - \frac{C_2}{\lambda}\cos\lambda t + \frac{C_2}{\lambda} \Rightarrow \dot q_1 = C_1\cos\lambda t + C_2\sin\lambda t$$
$$q_2(t) = \frac{D_1}{\lambda}\sin\lambda t - \frac{D_2}{\lambda}\cos\lambda t + \frac{D_2}{\lambda} \to \dot q_2 = D_1\cos\lambda t + D_2\sin\lambda t$$
$$\dot q_3 = q_1\dot q_2 - q_2\dot q_1$$

$$\dot q_3 = \left(\frac{C_1}{\lambda}\sin\lambda t - \frac{C_2}{\lambda}\cos\lambda t + \frac{C_2}{\lambda}\right)\left(D_1\cos\lambda t + D_2\sin\lambda t\right) - \left(\frac{D_1}{\lambda}\sin\lambda t - \frac{D_2}{\lambda}\cos\lambda t + \frac{D_2}{\lambda}\right)\left(C_1\cos\lambda t + C_2\sin\lambda t\right)$$

$$= \cancel{\frac12\left(\frac{C_1D_1}{\lambda} - \frac{C_2D_2}{\lambda}\right)}\sin 2\lambda t + \frac{C_1D_2}{\lambda}\sin^2\lambda t - \frac{C_2D_1}{\lambda}\cos^2\lambda t + \frac{C_2D_1}{\lambda}\cos\lambda t + \cancel{\frac{C_2D_2}{\lambda}\sin\lambda t}$$
$$- \cancel{\frac12\left(\frac{D_1C_1}{\lambda} - \frac{D_2C_2}{\lambda}\right)}\sin 2\lambda t - \frac{D_1C_2}{\lambda}\sin^2\lambda t + \frac{D_2C_1}{\lambda}\cos^2\lambda t - \frac{D_2C_1}{\lambda}\cos\lambda t - \cancel{\frac{D_2C_2}{\lambda}\sin\lambda t}$$

$$\dot q_3 = \frac1\lambda(C_1D_2 - D_1C_2)\sin^2\lambda t + \frac1\lambda(C_1D_2 - D_1C_2)\cos^2\lambda t + \frac1\lambda(C_2D_1 - D_2C_1)\cos\lambda t + \cancel{\frac1\lambda C_2D_2} -$$

$$\dot q_3 = \frac1\lambda(C_1D_2 - D_1C_2) - \frac1\lambda(C_1D_2 - D_1C_2)\cos\lambda t = \frac1\lambda(C_1D_2 - D_1C_2)(1-\cos\lambda t)$$

$$q_3(t) = \overset{0}{q_3(0)} + \frac1\lambda(C_1D_2 - D_1C_2)t - \frac1{\lambda^2}(C_1D_2 - D_1C_2)\sin\lambda t$$

$$\boxed{q_3(t) = \frac1{\lambda^2}(C_1D_2 - D_1C_2)(\lambda t - \sin\lambda t)} \sim \frac{(\lambda t)^3}{\lambda^2} = \lambda t^3$$

$$q_1(t) = \frac1\lambda C_1\sin\lambda t - \frac1\lambda C_2\cos\lambda t + \frac{C_2}{\lambda}$$
$$q_2(t) = \frac1\lambda D_1\sin\lambda t - \frac1\lambda D_2\cos\lambda t + \frac{D_2}{\lambda}$$

$$\left(q_1 - \frac{C_2}{\lambda}\right)^2 + \left(q_2 - \frac{D_2}{\lambda}\right)^2 = \frac{1}{\lambda^2} C_1^2 \sin^2\lambda t - \frac{1}{\lambda^2} C_1 C_2 \sin 2\lambda t + \frac{C_2^2}{\lambda^2}\cos^2\lambda t +$$

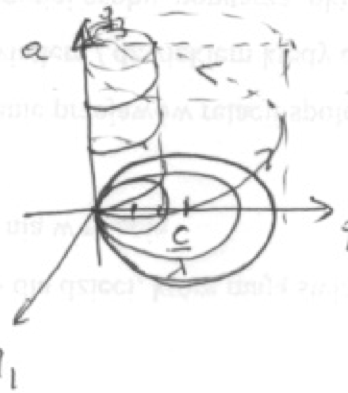
$$+ \frac{1}{\lambda^2} D_1^2 \sin^2\lambda t - \frac{1}{\lambda^2} D_1 D_2 \sin 2\lambda t + \frac{D_2^2}{\lambda^2}\cos^2\lambda t$$

$$= \frac{1}{\lambda^2}(C_1^2 + D_1^2)\sin^2\lambda t + \frac{1}{\lambda^2}(C_2^2 + D_2^2)\cos^2\lambda t - \frac{1}{\lambda^2}(C_1 C_2 + D_1 D_2)\sin 2\lambda t$$

$C_1 = D_2 = C$

gdyby $\cancel{C_1 = C_2,\ D_1 =}$ $C_2 = D_1 = 0$, to

$$q_1^2 + \left(q_2 - \frac{C}{\lambda}\right)^2 = \frac{1}{\lambda^2} C^2 \quad \text{– okrąg}$$



$$q_3(t) = \frac{1}{\lambda^2} C^2 (\lambda t - \sin\lambda t)$$

$$q_3(1) = \frac{1}{\lambda^2} C^2 (\lambda - \sin\lambda) = a \Rightarrow \underline{\underline{C}}$$

$\lambda = 2k\pi$ $\qquad \frac{C}{\lambda} = \pm\left(\frac{a}{\lambda - \sin\lambda}\right)^{1/2}$

$\sin\lambda = 0$ $\qquad \frac{C}{\lambda} = \pm\left(\frac{a}{\lambda}\right)^{1/2}$