Mechanika analityczna, II r AiR

Wykład 4

Elementy rachunku wariacyjnego

$\mathfrak{X}, \mathcal{Y}$ — p. Banacha $\quad f: \mathfrak{X} \to \mathcal{Y}$

$$\frac{\|f(x+v) - f(x) - Df(x)v\|}{\|v\|} \xrightarrow[v \to 0]{} 0 \qquad Df(x) \text{ — pochodna F.}$$

$$Df(x)v = \left.\frac{d}{d\alpha}\right|_{\alpha=0} f(x+\alpha v) \quad \text{— pochodna G.}$$

$\mathfrak{X}$ — przestrzeń krzywych $x(\cdot)$ na odcinku $[t_0, t_1] \to \mathbb{R}^n$

$$x(\cdot) = \{(t, x(t)) \mid t_0 \le t \le t_1\}$$

$\mathfrak{X}$ — p. liniowa $\quad (x_1(\cdot) + x_2(\cdot))(t) = x_1(t) + x_2(t) \in \mathbb{R}^n$

$$(\alpha x(\cdot))(t) = \alpha x(t) \in \mathbb{R}^n$$

$\mathfrak{X}$ — p. unormowana, jeżeli $x(\cdot) \in C^k$, to

$$\|x(\cdot)\|_k = \max_{t_0 \le t \le t_1} \|x(t)\| + \max_{t_0 \le t \le t_1} \|\dot x(t)\| + \dots + \max_{t_0 \le t \le t_1} \|x^{(k)}(t)\|$$

lub $\quad \|x(\cdot)\|_{L_k} = \left(\int_{t_0}^{t} \|x(t)\|^k \, dt \right)^{1/k}$

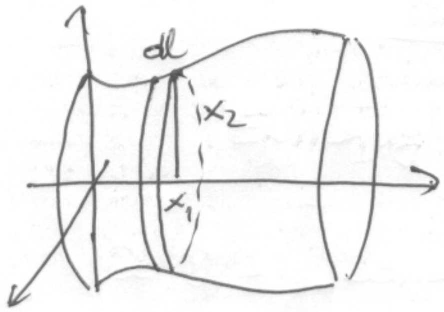
Funkcjonał: $\quad f: \mathfrak{X} \to \mathbb{R}$

Przykłady: $\quad \mathfrak{X}$ — krzywe w $\mathbb{R}^2$

1. $f(x(\cdot)) = \int_{t_0}^{t_1} \|\dot x(t)\| \, dt$ — długość krzywej $= \int_{t_0}^{t_1} \sqrt{\dot x_1^2(t) + \dot x_2^2(t)} \, dt$

2. $f(x(\cdot)) = \int_{t_0}^{t_1} x_2 \dot x_1 \, dt$ — pole pod krzywą

3. Pole figury obrotowej



$$2\pi x_2\,dl = 2\pi x_2 \sqrt{\dot{x}_1^2+\dot{x}_2^2}\,dt$$

$$f(x(\cdot)) = 2\pi \int_{t_0}^{t_1} x_2 \sqrt{\dot{x}_1^2+\dot{x}_2^2}\,dt$$

4. Średnia krzywizna^2 krzywej

$$f(x(\cdot)) = \int_{t_0}^{t_1} \frac{(\dot{x}_1\ddot{x}_2-\ddot{x}_1\dot{x}_2)^2}{(\dot{x}_1^2+\dot{x}_2^2)^3}\,dt$$

Rozważmy funkcjonały postaci:

$$f(x(\cdot)) = \int_{t_0}^{t_1} L(t, x(t), \dot{x}(t))\,dt \qquad L : \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}$$

$$x(t) = (x_1(t), x_2(t), \dots, x_n(t))$$

Obliczanie pochodnej: $x(\cdot), v(\cdot) \in \mathcal{X}$

$$Df(x(\cdot))v(\cdot) = \frac{d}{d\alpha}\Big|_{\alpha=0} \int_{t_0} L(t, x(t)+\alpha v(t), \dot{x}(t)+\alpha\dot{v}(t))\,dt =$$

$$= \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial x}(t,x,\dot{x})\,v(t) + \frac{\partial L}{\partial \dot{x}}(t,x,\dot{x})\,\dot{v}(t)\right)dt$$

Ale
$$\int_{t_0}^{t_1} \underset{u}{\frac{\partial L}{\partial \dot{x}}}\,\underset{\dot{v}}{\dot{v}}\,dt = \frac{\partial L}{\partial \dot{x}}v\Big|_{t_0}^{t_1} - \int_{t_0}^{t_1} \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\,v(t)\,dt$$

Niech $v(t_0)=v(t_1)=0$. Wówczas

$$Df(x(\cdot))v(\cdot) = \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right)v(t)\,dt.$$

Ekstremum funkcjonału:

$x(\cdot)$ jest minimum funkcjonału $f(x(\cdot))$, jeżeli

$$\forall v(\cdot) \qquad f(x(\cdot)+v(\cdot)) \geq f(x(\cdot))$$

Mamy $\quad f(x(\cdot)+v(\cdot)) = f(x(\cdot)) + Df(x(\cdot))v(\cdot) + h.o.t$

Dla małych $v(\cdot)$ $\quad$ ($\|v(\cdot)\|$ małe)

$$f(x+v(\cdot)) \cong f(x(\cdot)) + Df(x(\cdot))v(\cdot) \geq f(x(\cdot)),$$

~~Gdyby~~ Czyli $\quad Df(x(\cdot))v(\cdot) \geq 0$.

Gdyby $\quad Df(x(\cdot))v(\cdot) > 0$, wtedy przy zamianie $v(\cdot) \mapsto -v(\cdot)$ byłoby $Df(x(\cdot))v(\cdot) < 0$. Dlatego, warunek konieczny minimum (ekstremum) jest

$$Df(x(\cdot))v(\cdot) = 0 \qquad \forall v(\cdot) \quad \|v(\cdot)\| < \varepsilon$$

Punkt $x(\cdot)$ nazywamy p. stacjonarnym funkcjonału $f(x(\cdot))$, jeżeli $\quad Df(x(\cdot))v(\cdot) = 0 \quad \forall v(\cdot)$.

Warunek na p. stacjonarny funkcjonału

$$f(x(\cdot)) = \int_{t_0}^{t_1} L(t, x(t), \dot{x}(t))\,dt$$

podaje równanie Eulera-Lagrange'a

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \Rightarrow x(\cdot)$$

Przykłady:

1. Najkrótsze linie w $\mathbb{R}^2$: $x(t)=(x_1(t),x_2(t))$

$$f(x(\cdot)) = \int_{t_0}^{t_1} \sqrt{\dot{x}_1^2+\dot{x}_2^2}\, dt$$

Mamy $L(t,x,\dot{x}) = \sqrt{\dot{x}_1^2+\dot{x}_2^2}$, $\frac{\partial L}{\partial x}=0$, $\frac{\partial L}{\partial \dot{x}_1} = \frac{1}{2}\frac{2\dot{x}_1}{\sqrt{}}$,

$$\frac{\partial L}{\partial \dot{x}_1} = \frac{\dot{x}_1}{L}, \quad \frac{\partial L}{\partial \dot{x}_2} = \frac{\dot{x}_2}{L}$$

REL:
$$\left.\begin{aligned} \frac{d}{dt}\frac{\dot{x}_1}{L} = 0 &\Rightarrow \frac{\dot{x}_1}{L} = C_1 \\ \frac{d}{dt}\frac{\dot{x}_2}{L} = 0 &\Rightarrow \frac{\dot{x}_2}{L} = C_2 \end{aligned}\right\} \quad \frac{dx_2}{dx_1} = C$$

$x_2 = Cx_1 + D$ – linia prosta

$(x_2(t) = at+b,\ x_1(t) = t)$

2. Krzywa dająca ~~największe~~ najmniejsze pole p. bocznej figury obrotowej:

$$f(x(\cdot)) = 2\pi \int_{t_0}^{t_1} x_2\sqrt{\dot{x}_1^2+\dot{x}_2^2}\, dt \qquad x(t)=(x_1(t),x_2(t))$$

$L = x_2\sqrt{\dot{x}_1^2+\dot{x}_2^2}$ $\qquad \frac{\partial L}{\partial x_1} = 0$, $\frac{\partial L}{\partial x_2} = \sqrt{}$,

$$\frac{\partial L}{\partial \dot{x}_1} = \frac{\dot{x}_1 x_2}{\sqrt{}} \qquad \frac{\partial L}{\partial \dot{x}_2} = \frac{\dot{x}_2 x_2}{\sqrt{}}$$

REL: $\frac{d}{dt}\frac{\dot{x}_1 x_2}{\sqrt{}} = 0 \Rightarrow \frac{\dot{x}_1 x_2}{\sqrt{}} = C \Rightarrow \frac{\dot{x}_1 x_2}{C} = \sqrt{}$

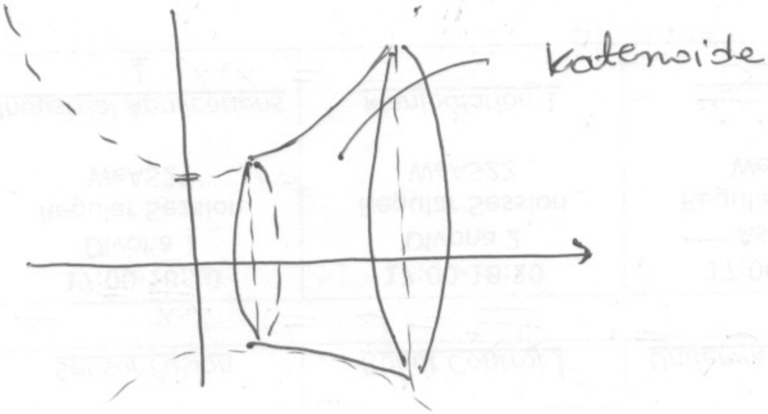
$$\frac{d}{dt}\frac{\dot{x}_2 x_2}{\sqrt{}} = \sqrt{}$$

$$C\frac{d}{dt}\frac{\dot{x}_2 x_2}{\dot{x}_1 x_2} = \frac{\dot{x}_1 x_2}{C} \Rightarrow C^2 \frac{d}{dt}\frac{dx_2}{dx_1} = \dot{x}_1 x_2$$

$$C^2 \frac{d^2 x_2}{dx_1^2}\dot{x}_1 = \dot{x}_1 x_2 \Rightarrow \dot{x}_1 = 0 \text{ – pik}$$

$$\frac{d^2 x_2}{dx_1^2} = \gamma^2 x_2 \Rightarrow \lambda_1 = -\gamma,\ \lambda = \gamma$$

$x_2(x_1) = A\,ch\,\gamma x_1 + B\,sh\,\gamma x_1 = D\,ch(\gamma x_1 + E)$ – Krzywa Łańcuchowa



Zadanie warunkowe: $x(\cdot) \in \mathcal{X}$

$$\min\left(f_1(x(\cdot))\right) = \int_{t_0}^{t_1} L(t, x(t), \dot{x}(t))\,dt,$$

warunek: $f_2(x(\cdot)) = \int_{t_0}^{t_1} K(t, x(t), \dot{x}(t))\,dt = a = const.$

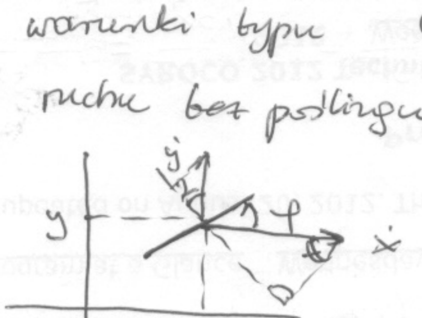
– zadanie izoperymetryczne

Dydone: znaleźć krzywą zamkniętą, o zadanym obwodzie, ograniczającą największe pole

$$\min\left(f(x(\cdot))\right) = \int_{t_0}^{t_1} L(t, x(t), \dot{x}(t))\,dt \quad p.w. \quad G(t, x(t), \dot{x}(t)) = 0 \ \forall t$$

– zadanie nieholonomiczne (Valerij Kozlov)

warunki typu $G(t, x(t), \dot{x}(t)) = 0$ pojawiają się przy ruchu bez poślizgu, Tyczkin Czaplygina



$$\dot{x}\sin\varphi - \dot{y}\cos\varphi = 0$$

znaleźć trajektorię T. Cz. odpowiadającą minimum energii;

Warunki:

Zadanie izoperymetryczne: tworzymy

$$\mathcal{L}(t,x,\dot{x},\lambda) = L(t,x,\dot{x}) + \lambda K(t,x,\dot{x}), \quad \lambda - \text{const.}$$

i piszemy REL dla $\mathcal{L}$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \mathcal{L}}{\partial x} = 0 \Rightarrow x(t,\lambda), \dot{x}(t,\lambda)$$

$$\int_{t_0}^{t_1} K(t, x(t,\lambda), \dot{x}(t,\lambda))\,dt = \alpha \Rightarrow \lambda$$

Zadanie nieholonomiczne: tworzymy

$$\mathcal{L}(t,x,\dot{x},\lambda) = L(t,x,\dot{x}) + \lambda(t)\, G(t,x,\dot{x}), \quad \lambda \text{ funkcja czasu}$$

i piszemy REL dla $\mathcal{L}$:

Ex: Zadanie Dydony: $x(t) = (x_1(t), x_2(t))$

$$\max \int_{t_0}^{t_1} x_2 \dot{x}_1\,dt, \qquad \int_{t_0}^{t_1} \sqrt{\dot{x}_1^2 + \dot{x}_2^2}\,dt = l$$

$$L = x_2\dot{x}_1, \quad K = \sqrt{\dot{x}_1^2+\dot{x}_2^2}, \quad \mathcal{L} = x_2\dot{x}_1 + \lambda\sqrt{\dot{x}_1^2+\dot{x}_2^2}$$

REL: $\frac{\partial \mathcal{L}}{\partial x_1} = 0$, $\frac{\partial \mathcal{L}}{\partial x_2} = \dot{x}_1$, $\frac{\partial \mathcal{L}}{\partial \dot{x}_1} = x_2 + \frac{\lambda \dot{x}_1}{\sqrt{\ }}$, $\frac{\partial \mathcal{L}}{\partial \dot{x}_2} = \frac{\lambda \dot{x}_2}{\sqrt{\ }}$

$$\frac{d}{dt}\left(x_2 + \frac{\lambda\dot{x}_1}{\sqrt{\ }}\right) = 0 \Rightarrow x_2 + \frac{\lambda\dot{x}_1}{\sqrt{\ }} = C.$$

$$\frac{d}{dt}\frac{\lambda\dot{x}_2}{\sqrt{\ }} = \dot{x}_1 \Rightarrow \frac{\lambda\dot{x}_2}{\sqrt{\ }} = x_1 + D$$

$$\frac{\lambda}{\sqrt{\ }} = \frac{C-x_2}{\dot{x}_1}; \quad \frac{\dot{x}_2(C-x_2)}{\dot{x}_1} = x_1 + D \qquad \frac{dx_2}{dx_1}(C-x_2) = x_1 + D$$

$$(C-x_2)\,dx_2 = (x_1+D)\,dx_1$$

$$(x_2-C)\,dx_2 + (x_1+D)\,dx_1 = 0 \Rightarrow \frac{1}{2}(x_2-C)^2 + \frac{1}{2}(x_1+D)^2 = \frac{1}{2}E^2$$

$$(x_1+D)^2 + (x_2-C)^2 = E^2$$