Mechanika analityczna, II r AiR

Wykład 3

Różniczkowanie

$\mathcal{X}$ – p. liniowa nad $\mathbb{R}$

$$x, y \in \mathcal{X} \Rightarrow x+y \in \mathcal{X}$$
$$x \in \mathcal{X}, \alpha \in \mathbb{R} \Rightarrow \alpha x \in \mathcal{X}$$

norma : $\| \; \| : \mathcal{X} \to \mathbb{R}$

$\|x\| \geq 0$, $\|x\| = 0 \Leftrightarrow x = 0$ – dodatniość

$\|\alpha x\| = |\alpha| \|x\|$ – bezwzględna ~~liniowość~~ jednorodność

$\|x+y\| \leq \|x\| + \|y\|$ – n. trójkąta

zupełność : $x_n \in \mathcal{X} \Rightarrow \lim_{n \to +\infty} x_n \in \mathcal{X}$

$\mathcal{X}$ jest p. Banacha $\qquad L : \mathcal{X} \to \mathbb{R}$ liniowe, jeżeli
$L(\alpha x + \beta y) = \alpha L(x) + \beta L(y)$

$f : \mathcal{X} \to \mathbb{R}$

Pochodna $Df(x) : \mathcal{X} \to \mathbb{R}$, takie że $Df(x)$ jest liniowe i
$$\frac{\| f(x+v) - f(x) - Df(x)v \|}{\|v\|} \underset{v \to 0}{\longrightarrow} 0$$

Jeżeli f jest różniczkowalna, to
$$Df(x)v = \left.\frac{d}{d\alpha}\right|_{\alpha=0} f(x+\alpha v)$$

Ex: $f : \mathbb{R}^n \to \mathbb{R} \qquad f(x) = x^T Q x$

$$Df(x)v = \left.\frac{d}{d\alpha}\right|_{\alpha=0} (x+\alpha v)^T Q (x+\alpha v) = v^T Q x + x^T Q v =$$
$$= x^T Q v + x^T Q^T v = x^T(Q+Q^T)v$$

$$D(x^T Q x) = x^T(Q+Q^T)$$

Ex: $f : Mat(n) \to \mathbb{R}$ $\quad f(X) = \operatorname{tr} X^T X$

$$Df(X)H = \frac{d}{d\alpha}\Big|_{\alpha=0} \operatorname{tr} (X+\alpha H)^T (X+\alpha H) = \operatorname{tr} H^T X + \operatorname{tr} X^T H =$$

$$= \operatorname{tr} X^T H + \operatorname{tr} (X^T H)^T = 2 \operatorname{tr} X^T H$$

$$D \operatorname{tr} X^T X = 2 \operatorname{tr} X^T.$$

$f(X) = \det X$

$$Df(X)H = \frac{d}{d\alpha}\Big|_{\alpha=0} \det (X+\alpha H) = \frac{d}{d\alpha}\Big|_{\alpha=0} \det [X_1+\alpha H_1, X_2+\alpha H_2, \dots, X_n+\alpha H_n] =$$

$$= \det[H_1, X_2, \dots, X_n] + \det[X_1, H_2, \dots, X_n] + \dots + \det[X_1, \dots, H_n]$$

$\mathcal{X}$ – p. liniowe nad $\mathbb{R}$

iloczyn skalarny = ~~forma~~ $(\ ,\ ) : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$

$(x,x) \geq 0$, $\cancel{(\alpha x, y) = \alpha (x,y)}$, $\cancel{(\alpha x+\beta y, z) = \alpha (x,z) + \beta (y,z)}$

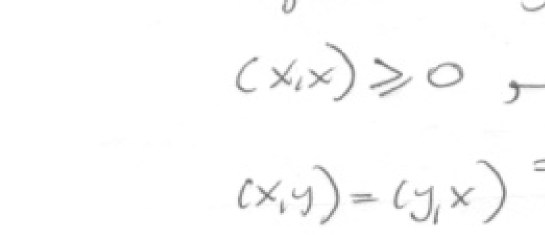
$(x,y) = (y,x)$

$(\alpha x + \beta y, z) = \alpha (x,z) + \beta (y,z)$

$\mathcal{X}$ zupełna z $(\ ,\ )$ – p. Hilberta

Każde p. Hilberta jest Banacha, bo $\|x\| = (x,x)^{1/2}$.

$\mathcal{X}$ – przestrzeń krzywych zdefiniowanych na odcinku $[t_0, t_1]$



$x(\cdot) = \{(t, x(t)) \mid t_0 \leq t \leq t_1\}$ $\quad x(t) \in \mathbb{R}^n$

$y(\cdot) = \{(t, y(t)) \mid t_0 \leq t \leq t_1\}$

$(x(\cdot) + y(\cdot))(t) = x(t) + y(t)$

$(\alpha x(\cdot))(t) = \alpha x(t)$

$\mathcal{X}$ – p. liniowa

normy: załóżmy, że krzywe są różniczkowalne

$$\|x(\cdot)\|_k = \max_t \|x(t)\| + \max_t \|\dot{x}(t)\| + \dots + \max \|x^{(k)}(t)\|.$$

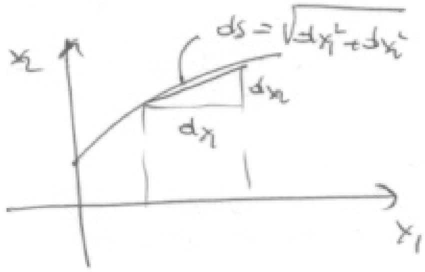
Na ogół nie mamy tu ~~ciągł~~ zupełności. Ale

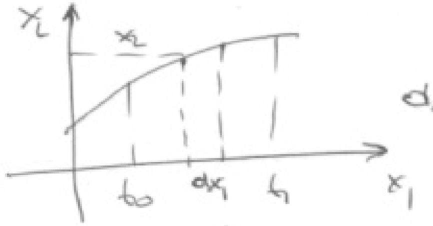
$$\|x(\cdot)\|_{L_2} = \left(\int_{t_0}^{t_1} \|x(t)\|^2 dt \right)^{1/2},$$ to mamy p. zupełną (lub L_p)

Przykłady funkcjonałów $f: \mathcal{X} \to \mathbb{R}$

1). długość krzywej $f(x(\cdot)) = \left(\int_{t_0}^{t_1} \|\dot{x}(t)\|^{\#} dt \right)^{1/2} \left(\int_{t_0}^{t_1} \sqrt{\dot{x}_1^2 + \dot{x}_2^2}\, dt \right)$

2). pole pod krzywą $f(x(\cdot)) = \int_{t_0}^{t_1} x_2 \dot{x}_1 \, dt$

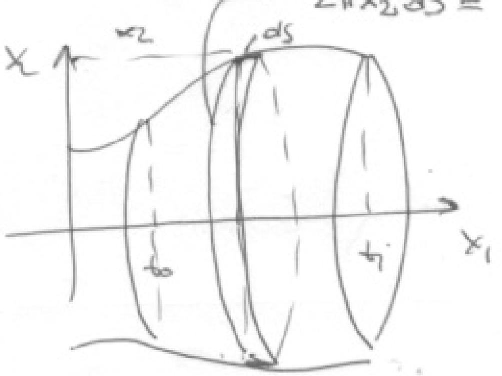




$dS = x_2 dx_1$

3). pole figury obrotowej

$2\pi x_2 ds = 2\pi x_2 \sqrt{\dot{x}_1^2 + \dot{x}_2^2}\, dt$



$$f(x(\cdot)) = 2\pi \int_{t_0}^{t_1} x_2 \sqrt{\dot{x}_1^2 + \dot{x}_2^2}\, dt$$

$$f(x(\cdot)) = \int_{t_0}^{t_1} \frac{(\dot{x}_1 \ddot{x}_2 - \ddot{x}_1 \dot{x}_2)^2}{(\dot{x}_1^2 + \dot{x}_2^2)^3} dt$$

krzywizna średniokwadratowa

Będziemy rozważać funkcjonały postaci

$$f(x(\cdot)) = \int_{t_0}^{t_1} L(t, x(t), \dot{x}(t))\, dt \quad , x(\cdot) \in \mathcal{X}$$

Pochodna: $v(\cdot) \in \mathcal{X}$

$$Df(x(\cdot))v(\cdot) = \frac{d}{d\alpha}\Big|_{\alpha=0} \int_{t_0}^{t_1} L(t, x(t)+\alpha v(t), \dot{x}(t)+\alpha\dot{v}(t))\,dt$$

$$= \int_{t_0}^{t_1} \frac{\partial L(t,x,\dot{x})}{\partial x} v(t)\,dt + \int_{t_0}^{t_1} \frac{\partial L(t,x,\dot{x})}{\partial \dot{x}} \dot{v}(t)\,dt$$

$$\int u\dot{v}\,dt = \cancel{\,} uv - \int \dot{u}v\,dt$$

$$\int \frac{\partial L}{\partial \dot{x}} \dot{v}\,dt = \frac{\partial L}{\partial \dot{x}} v - \int \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} v\,dt$$

Wiemy $v(t_0) = v(t_1) = 0$

$$Df(x(\cdot))v(\cdot) = \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right) v\,dt.$$

Warunek na ekstremum:

Minimum $x^*(\cdot)$, jeżeli $\forall u^*(\cdot)$ $\quad f(x^*(\cdot)+u^*(\cdot)) \geqslant f(x^*(\cdot))$

$$f(x^*(\cdot)+u^*(\cdot)) \cong f(x^*(\cdot)) + Df(x^*(\cdot))u^*(\cdot) \geqslant f(x^*(\cdot))$$

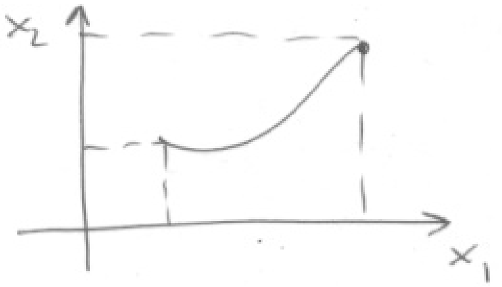
$$\Rightarrow \quad Df(x^*(\cdot))u^*(\cdot) \geqslant 0$$

ale $u^*(\cdot)$ musi być równe, jeżeli
$Df(x^*(\cdot))u(\cdot) > 0$, to $Df(x^*(\cdot))(-u(\cdot)) < 0$,
trzeba zatem by $\underline{Df(x^*(\cdot)) = 0}$.

Równanie Eulera-Lagrange'a:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 .$$

Ex: Najkrótsza linia



$$L(t, x, \dot{x}) = \|\dot{x}(t)\| = \sqrt{\dot{x}_1^2 + \dot{x}_2^2}$$

$$\frac{\partial L}{\partial \dot{x}_1} = \frac{1}{2\sqrt{}} \cdot 2\dot{x}_1 = \frac{\dot{x}_1}{\sqrt{}}, \quad \frac{\partial L}{\partial x_1} = 0$$

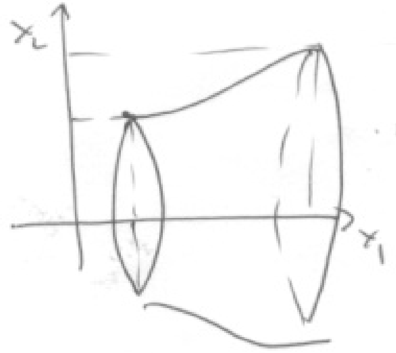
$$\frac{\partial L}{\partial \dot{x}_2} = \frac{1}{2\sqrt{}} 2\dot{x}_2 = \frac{\dot{x}_2}{\sqrt{}}, \quad \frac{\partial L}{\partial x_2} = 0$$

Mamy zatem:

$$\left.\begin{aligned} \frac{\dot{x}_1}{\sqrt{}} &= c_1 \\ \frac{\dot{x}_2}{\sqrt{}} &= c_2 \end{aligned}\right\} \quad \frac{dx_2}{dx_1} = C$$

$$\underline{x_2 = Cx_1 + d}$$ linia prosta

Ex: Krzywa dająca najmniejsze pole pow. bocznej



$$L(t, x, \dot{x}) = x_2 \sqrt{\dot{x}_1^2 + \dot{x}_2^2}$$

$$\frac{\partial L}{\partial \dot{x}_1} = x_2 \frac{\dot{x}_1}{\sqrt{}} \qquad \frac{\partial L}{\partial x_1} = 0$$

$$\frac{\partial L}{\partial \dot{x}_2} = x_2 \frac{\dot{x}_2}{\sqrt{}} \qquad \frac{\partial L}{\partial x_2} = \sqrt{}$$

$$\begin{cases} \dfrac{x_2 \dot{x}_1}{\sqrt{}} = C \Rightarrow \dfrac{x_2}{\sqrt{}} = \dfrac{C}{\dot{x}_1} \\ \dfrac{d}{dt} \dfrac{x_2 \dot{x}_2}{\sqrt{}} - \sqrt{} = 0 \end{cases}$$

$$\frac{d}{dt}\frac{C\dot{x}_2}{\dot{x}_1} = \sqrt{\dot{x}_1^2+\dot{x}_2^2} = \frac{x_2\dot{x}_1}{C}$$

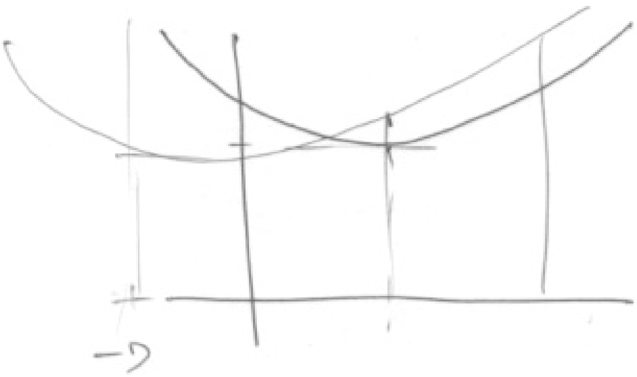
$$\frac{d}{dt}\frac{dx_2}{dx_1} = x_2\dot{x}_1 a^2$$

$$\frac{d^2x_2}{dx_1^2}\cdot\dot{x}_1 = a^2x_2\dot{x}_1 \Rightarrow \quad \dot{x}_1 = 0 \text{, postyk}$$

$$\frac{d^2x_2}{dx_1^2} = a^2x_2 \text{ – r. liniowe}$$

r. charakterystyczne $r^2 = a^2 \Rightarrow r_{1,2} = \pm a$

$x_2 = C_1e^{ax_1} + C_2e^{-ax_1}$ lub $\underline{x_2 = C\,\mathrm{ch}(ax_1 + D)}$



– krzywe Taricadowe, figura = katenoida

Ekstrema warunkowe:

$$\min f(x(\cdot)) = \int_{t_0}^{t_1} L(t,x,\dot{x})dt \quad \text{przy warunku} \quad \int_{t_0}^{t_1} K(t,x,\dot{x})dt = a \text{ – dane}$$

} zadanie izoperymetryczne

Przykład: ~~Problem~~ Dydony, znaleźć ~~figur~~ krzywą zamkniętą o zadanej długości ograniczającą największe pole

$$\min f(x(\cdot)) = \int_{t_0}^{t_1} L(t,x,\dot{x})dt \quad \text{przy warunku} \quad G(t,x(t),\dot{x}(t)) = 0$$

– zadanie wakonomiane (Valerij Kozlov)

$L = \cancel{\sqrt{\sqrt{\dot{x}_1^2+\dot{x}_2}}}\ x_2 \dot{x}_1$

$K = \sqrt{\dot{x}_1^2+\dot{x}_2^2}$

Nowy lagranżian

$\mathcal{L} = L + \lambda K$

$\mathcal{L} = x_2\dot{x}_1 + \lambda\sqrt{\dot{x}_1^2+\dot{x}_2^2}$

$\frac{\partial \mathcal{L}}{\partial \dot{x}_1} = \lambda \frac{\dot{x}_1}{\sqrt{}} + x_2, \quad \frac{\partial \mathcal{L}}{\partial \dot{x}_2} = \lambda \frac{\dot{x}_2}{\sqrt{}}, \quad \frac{\partial \mathcal{L}}{\partial x_1} = 0, \quad \frac{\partial \mathcal{L}}{\partial x_2} = \dot{x}_1$

$\cancel{\frac{d}{dt}}\left(x_2 + \frac{\lambda \dot{x}_1}{\sqrt{}} \right) = c$

$\frac{d}{dt}\, \lambda \frac{\dot{x}_2}{\sqrt{}} = \dot{x}_1$

$$\left.\begin{aligned} \frac{\lambda \dot{x}_1}{\sqrt{}} &= c - x_2 \\ \frac{\lambda \dot{x}_2}{\sqrt{}} &= x_1 + d \end{aligned}\right\} \qquad \frac{dx_2}{dx_1} = \frac{d+x_1}{c-x_2}$$

$\cancel{(c-x_2)dx_2}$

$(x_2 - c)\,dx_2 + (x_1+d)\,dx_1 = 0$

$\frac{1}{2}x_2^2 - cx_2 + \frac{1}{2}x_1^2 + dx_1 \cancel{\pm \frac{1}{2}e}$

$(x_2-c)^2 + (x_1+d)^2 = R^2$ – okrąg

Inny funkcjonał

$$f(x(\cdot)) = \int_{t_0}^{t_1} L(t, x, \dot{x}, \ddot{x})\,dt$$

pochodna:

$$Df(x(\cdot))v(\cdot) = \frac{d}{d\alpha}\Big|_{\alpha=0} \int_{t_0}^{t_1} L(t, x+\alpha v(t), \dot{x}+\alpha\dot{v}(t), \ddot{x}+\alpha\ddot{v}(t))\,dt =$$

$$= \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial x} v + \frac{\partial L}{\partial \dot{x}} \dot{v} + \frac{\partial L}{\partial \ddot{x}} \ddot{v}\right) dt =$$

$$\int_{t_0}^{t_1} \frac{\partial L}{\partial \ddot{x}} \ddot{v}\, dt \;\Bigg|_{\substack{\ddot{v} = \dot{p} \quad p = \dot{v} \\ \frac{\partial L}{\partial \ddot{x}} = r \quad \dot{r} = \frac{d}{dt}\frac{\partial L}{\partial \ddot{x}}}} = \frac{\partial L}{\partial \ddot{x}} \dot{v}\Big|_{t_0}^{t_1} - \int_{t_0}^{t_1} \frac{d}{dt}\frac{\partial L}{\partial \ddot{x}} \dot{v}\, dt$$

$\dot{v}(t_0) = \dot{v}(t_1) = 0$ $\qquad$ $v(t_0) = v(t_1) = 0$

$$= \int_{t_0}^{t_1} \frac{\partial L}{\partial x} v + \left(\frac{\partial L}{\partial \dot{x}} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{x}}\right)\dot{v}\, dt = \int_{t_0}^{t_1} \frac{\partial L}{\partial x} v - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{x}}\right) v\, dt$$

Warunek konieczny ekstremum:

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{x}} = 0$$

Przykłady:

Ex: Krzywe o najmniejszym przyspieszeniu $x(t)=(x_1(t),x_2(t))$

$$f(x(\cdot)) = \int_{t_0}^{t_1} \|\ddot{x}(t)\|^2 \, dt = \int_{t_0}^{t_1} (\ddot{x}_1^2+\ddot{x}_2^2)\, dt$$

$$L(t,x,\dot{x},\ddot{x}) = \ddot{x}_1^2+\ddot{x}_2^2$$

$$\frac{\partial L}{\partial \ddot{x}_1} = 2\ddot{x}_1 \qquad \frac{\partial L}{\partial \ddot{x}_2} = 2\ddot{x}_2 \qquad \frac{\partial L}{\partial \dot{x}_1} = \frac{\partial L}{\partial x_1} = \frac{\partial L}{\partial \dot{x}_2} = \frac{\partial L}{\partial x_2} = 0$$

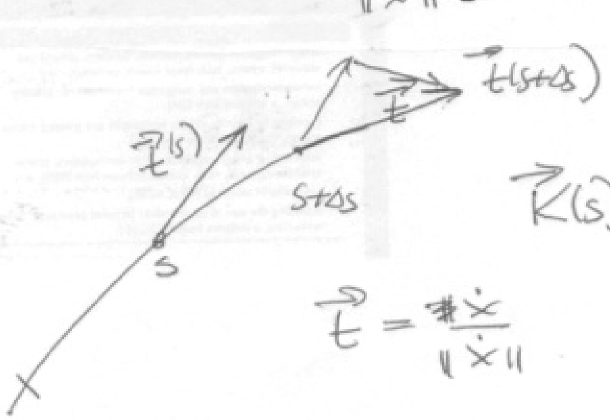
Zatem: $x_1^{(4)} = 0$, $x_2^{(4)} = 0$

$$x_1(t) = a_1 t^3 + a_2 t^2 + a_3 t + a_4$$
$$x_2(t) = b_1 t^3 + b_2 t^2 + b_3 t + b_4$$

Ex: Euler elastica $x(t)=(x_1(t),x_2(t))$

- krzywe o zadanej długości, ale najmniejszej krzywiźnie

$$k^2 = \frac{\|\dot{x}\|^2\|\ddot{x}\|^2-(\dot{x},\ddot{x})^2}{\|\dot{x}\|^6}$$



$$\vec{K}(s) = \lim_{\Delta s\to 0} \frac{\vec{t}(s+\Delta s)-\vec{t}(s)}{\Delta s} = \frac{d\vec{t}(s)}{ds}$$

$$\vec{t} = \frac{\dot{x}}{\|\dot{x}\|}$$

$$K(s) = \|\vec{k}(s)\| = \left\|\frac{d\vec{t}(s)}{ds}\right\|$$

$$ds = \|\dot{x}\|\, dt$$

$$\min \int_{t_0}^{t_1} K^2\,dt \qquad \int_{t_0}^{t_1} \|\dot{x}\|\,dt = \text{Konst}$$

$$K^2 = \frac{(\dot{x}_1^2+\dot{x}_2^2)(\ddot{x}_1^2+\ddot{x}_2^2) - (\dot{x}_1\ddot{x}_1+\dot{x}_2\ddot{x}_2)^2}{(\dot{x}_1^2+\dot{x}_2^2)^3}$$

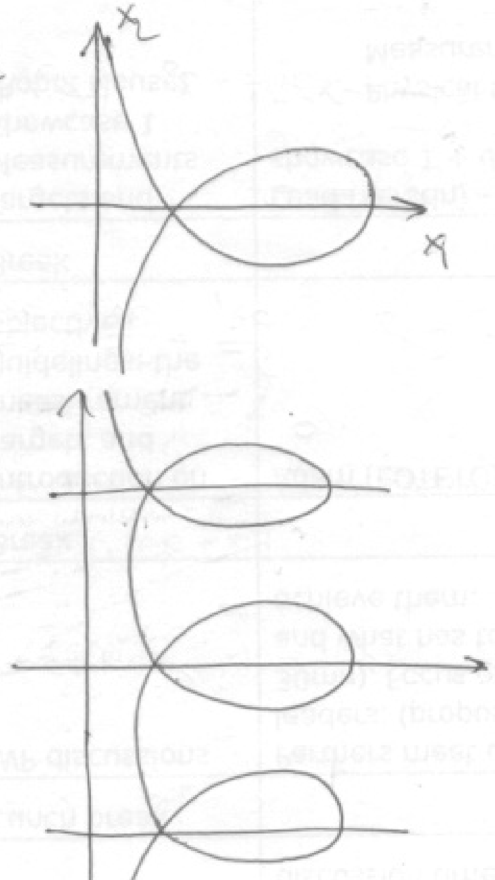
$$K^2 = \frac{\cancel{\dot{x}_1^2\ddot{x}_1^2} + \dot{x}_1^2\ddot{x}_2^2 + \dot{x}_2^2\ddot{x}_1^2 + \cancel{\dot{x}_2^2\ddot{x}_2^2} - \cancel{\dot{x}_1^2\ddot{x}_1^2} - 2\dot{x}_1\dot{x}_2\ddot{x}_1\ddot{x}_2 - \cancel{\dot{x}_2^2\ddot{x}_2^2}}{(\dot{x}_1^2+\dot{x}_2^2)^3}$$

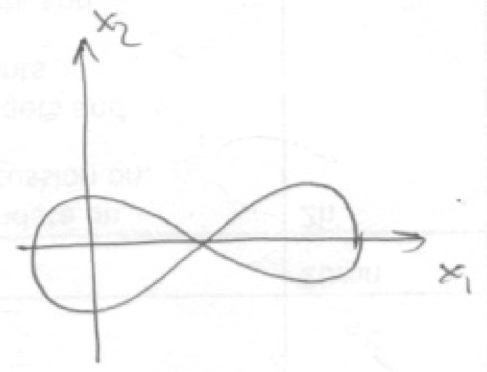
$$K^2 = \frac{(\dot{x}_1\ddot{x}_2 - \dot{x}_2\ddot{x}_1)^2}{(\dot{x}_1^2+\dot{x}_2^2)^3}$$

Problem

$$\min \int_{t_0}^{t_1} \frac{(\dot{x}_1\ddot{x}_2 - \dot{x}_2\ddot{x}_1)^2}{(\dot{x}_1^2+\dot{x}_2^2)^3} \quad \text{przy zadanym} \quad \int_{t_0}^{t_1} \sqrt{\dot{x}_1^2+\dot{x}_2^2}\,dt$$

Elastica





L. Euler

Modus inveniendi....

Lausannae et Genevae

1744.

Przy $x_1 = t = x$, $x_2 = y$

mamy

$$K^2 = \frac{y''^2_x}{\left(\sqrt{1+y'^2_x}\right)^3} \qquad \int_{x_0}^{x_1} \sqrt{1+y'^2_x}\,dx = L$$