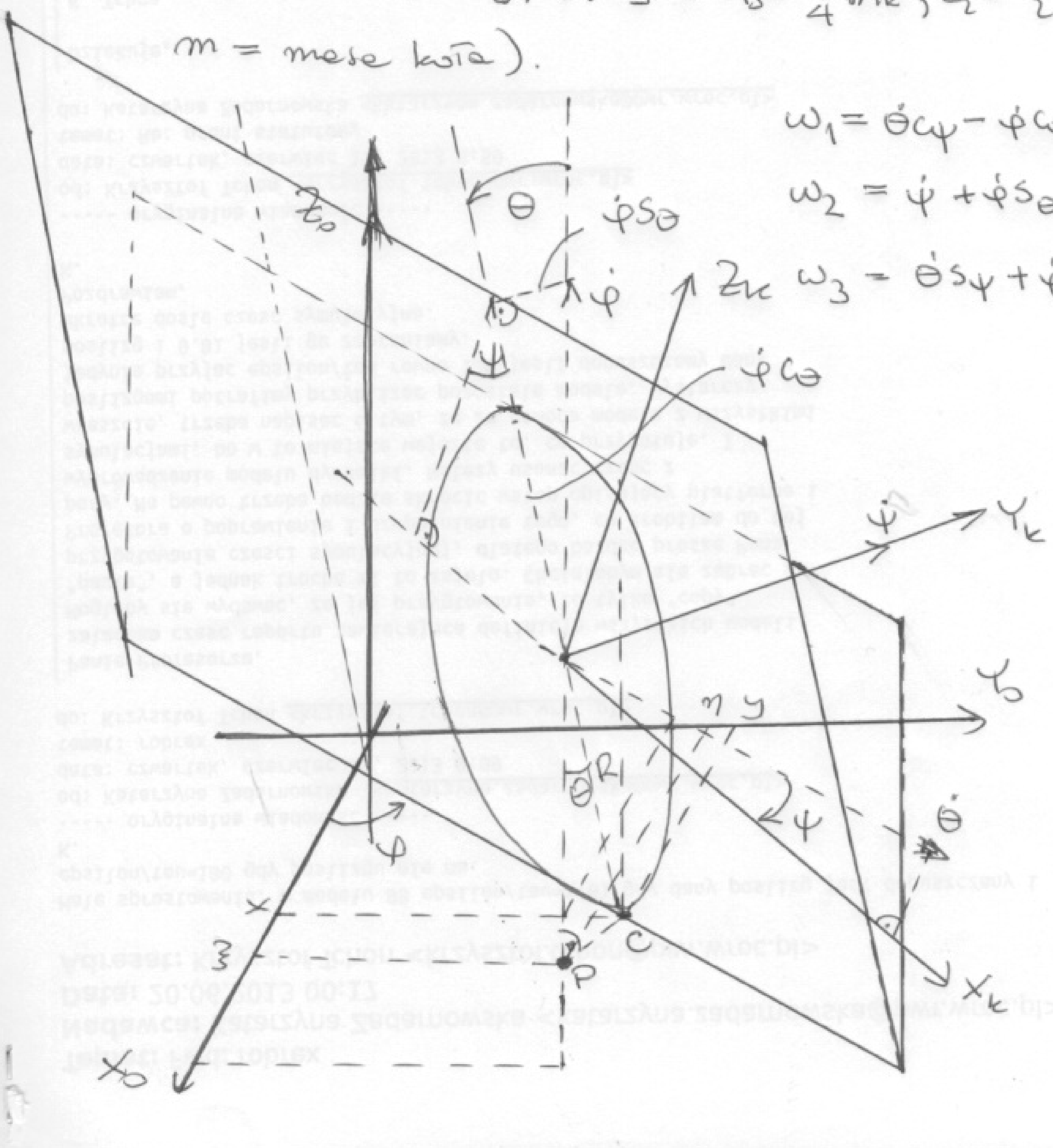
Mechanika analityczna, II r AiR

Wykład 11

Dynamika koła pochylonego i kuli

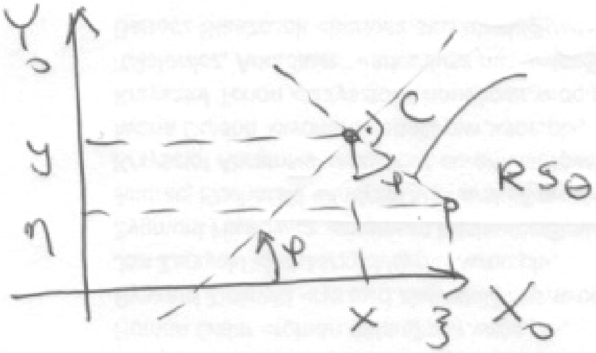
Badamy toczenie się koła przedstawionego na rysunku. Promień koła $= R$, momenty bezwładności względem osi x_k, y_k, z_k wynoszą, odpowiednio, I_3, I_2, I_3 ($I_3 = \frac{1}{4}mR^2$, $I_2 = \frac{1}{2}mR^2$, m = masa koła).



$$\omega_1 = \dot\theta c_\psi - \dot\varphi c_\theta s_\psi$$

$$\omega_2 = \dot\psi + \dot\varphi s_\theta$$

$$\omega_3 = \dot\theta s_\psi + \dot\varphi c_\theta c_\psi$$



$$q = (x, y, \varphi, \theta, \psi)$$

$$\dot q = (\dot x, \dot y, \dot\varphi, \dot\theta, \dot\psi)$$

$$\xi = x + R s_\theta s_\varphi, \quad \eta = y - R s_\theta c_\varphi, \quad \zeta = R c_\theta$$

$$L = K - V, \quad K = \tfrac{1}{2} I_B(\omega_1^2 + \omega_3^2) + \tfrac{1}{2} I_2 \omega_2^2 + \tfrac{1}{2} m v^2$$

$$v^2 = \dot\xi^2 + \dot\eta^2 + \dot\zeta^2, \quad V = m g R c_\theta$$

$$\dot\xi = \dot x + R\dot\theta c_\theta s_\varphi + R\dot\varphi s_\theta c_\varphi, \quad \dot\eta = \dot y - R\dot\theta c_\theta c_\varphi + R\dot\varphi s_\theta s_\varphi$$

$$\dot\zeta = -R\dot\theta s_\theta$$

$$v^2 = \dot x^2 + \dot y^2 + R^2\dot\theta^2 + R^2\dot\varphi^2 s_\theta^2 + 2\dot x\dot\theta R c_\theta s_\varphi + 2\dot x\dot\varphi R s_\theta c_\varphi - 2\dot y\dot\theta R c_\theta c_\varphi + 2\dot y\dot\varphi R s_\theta s_\varphi$$

alternatywnie

$$\cancel{v^2 = \ldots}(\dot x + R\dot\theta c_\theta s_\varphi)$$

$$v^2 = (\dot x + R\dot\varphi s_\theta c_\varphi)^2 + (\dot y + R\dot\varphi s_\theta s_\varphi)^2 + R^2\dot\theta^2 + 2R\dot\theta c_\theta(\dot x s_\varphi - \dot y c_\varphi)$$

$$\omega_1^2 + \omega_3^2 = \dot\theta^2 + \dot\varphi^2 c_\theta^2 + (\dot\psi + \dot\varphi s_\theta)^2$$

Ostatecznie

$$\boxed{L = \tfrac{1}{2} I_B(\dot\theta^2 + \dot\varphi^2 c_\theta^2) + \tfrac{1}{2} I_2(\dot\psi + \dot\varphi s_\theta)^2 + \tfrac{1}{2} m(\dot x + R\dot\varphi s_\theta c_\varphi)^2 + \tfrac{1}{2} m(\dot y + R\dot\varphi s_\theta s_\varphi)^2 + \tfrac{1}{2} m R^2\dot\theta^2 + m R\dot\theta c_\theta(\dot x s_\varphi - \dot y c_\varphi) - m g R c_\theta}$$

Dla $\theta=0$ otrzymujemy

$$L = \frac{1}{2} I_B \dot{\varphi}^2 + \frac{1}{2} I_2 \dot{\psi}^2 + \frac{1}{2} m(\dot{x}^2+\dot{y}^2),$$ jak w przypadku

koła toczącego się pionowo.

Ograniczenia: brak poślizgu wzdłużnego i poprzecznego

$$A(\varphi) = \begin{bmatrix} s_\varphi, & -c_\varphi, & 0, & 0, & 0 \\ c_\varphi, & s_\varphi, & 0, & 0, & -R \end{bmatrix}$$

Mamy $G(\varphi) = \begin{bmatrix} Rc_\varphi & 0 & 0 \\ Rs_\varphi & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, tzn.

$$\dot{x} = R\eta_1 c_\varphi, \quad \dot{y} = R\eta_1 s_\varphi, \quad \dot{\varphi} = \eta_2, \quad \dot{\theta} = \eta_3, \quad \dot{\psi} = \eta_1$$

Równania dynamiki, wobec powyższych, przyjmują:

$$\begin{cases} mR\ddot{x}c_\varphi + mR\ddot{y}s_\varphi + (I_2+mR^2)\ddot{\varphi}s_\theta + I_2\ddot{\psi} + (I_2+2mR^2)\dot{\varphi}\dot{\theta}c_\theta = 0 \\ mR\ddot{x}s_\theta c_\varphi + mR\ddot{y}s_\theta s_\varphi + (mR^2 s_\theta^2 + I_B c_\theta^2 + I_2 s_\theta^2)\ddot{\varphi} + I_2\ddot{\psi}s_\theta + \\ \quad + 2(mR^2+I_2-I_B)\dot{\varphi}\dot{\theta}s_\theta c_\theta - I_2\dot{\theta}\dot{\psi}c_\theta = 0 \\ mR(\ddot{x}s_\varphi - \ddot{y}c_\varphi)c_\theta + (I_B+mR^2)\ddot{\theta} - (mR^2+I_2-I_B)\dot{\varphi}^2 s_\theta c_\theta - \\ \quad - I_2\dot{\varphi}\dot{\psi}c_\theta + mgRs_\theta = 0. \end{cases}$$

Redukcja:

$$\ddot{x} = R\dot{\eta}_1 c_\varphi - R\eta_1\dot{\varphi} s_\varphi,\quad \ddot{y} = R\dot{\eta}_1 s_\varphi + R\eta_1\dot{\varphi} c_\varphi,\quad \ddot{\varphi} = \dot{\eta}_2,\quad \ddot{\theta} = \dot{\eta}_3,$$

$$\ddot{\psi} = \dot{\eta}_1$$

Otrzymujemy:

$$\underline{mR^2\dot{\eta}_1} c^2_\varphi - mR^2 \cancel{\eta_1\dot{\varphi} s_\varphi c_\varphi} + \underline{mR^2\dot{\eta}_1} s^2_\varphi + mR^2\cancel{\eta_1\dot{\varphi} c_\varphi s_\varphi} + (I_2+mR^2)\dot{\eta}_2 s_\theta +$$

$$+ \underline{I_2\dot{\eta}_1} + (I_2+2mR^2)\eta_2\eta_3 c_\theta = 0$$

(1) $$\boxed{(mR^2+I_2)\dot{\eta}_1 + (mR^2+I_2)s_\theta\dot{\eta}_2 + (I_2+2mR^2)\eta_2\eta_3 c_\theta = 0}$$

$$\underline{\dot{\eta}_1 mR^2 s_\theta} c^2_\varphi - mR^2\cancel{\eta_1\dot{\varphi} s_\theta s_\varphi c_\varphi} + \underline{mR^2\dot{\eta}_1 s_\theta} s^2_\varphi + mR^2\cancel{\eta_1\dot{\varphi} s_\theta s_\varphi c_\varphi} +$$

$$+ (mR^2 s^2_\theta + I_{13} c^2_\theta + I_2 s^2_\theta)\dot{\eta}_2 + \underline{I_2\dot{\eta}_1 s_\theta} + 2(mR^2+I_2-I_{13})\eta_2\eta_3 s_\theta c_\theta -$$

$$- I_2\eta_1\eta_3 c_\theta = 0$$

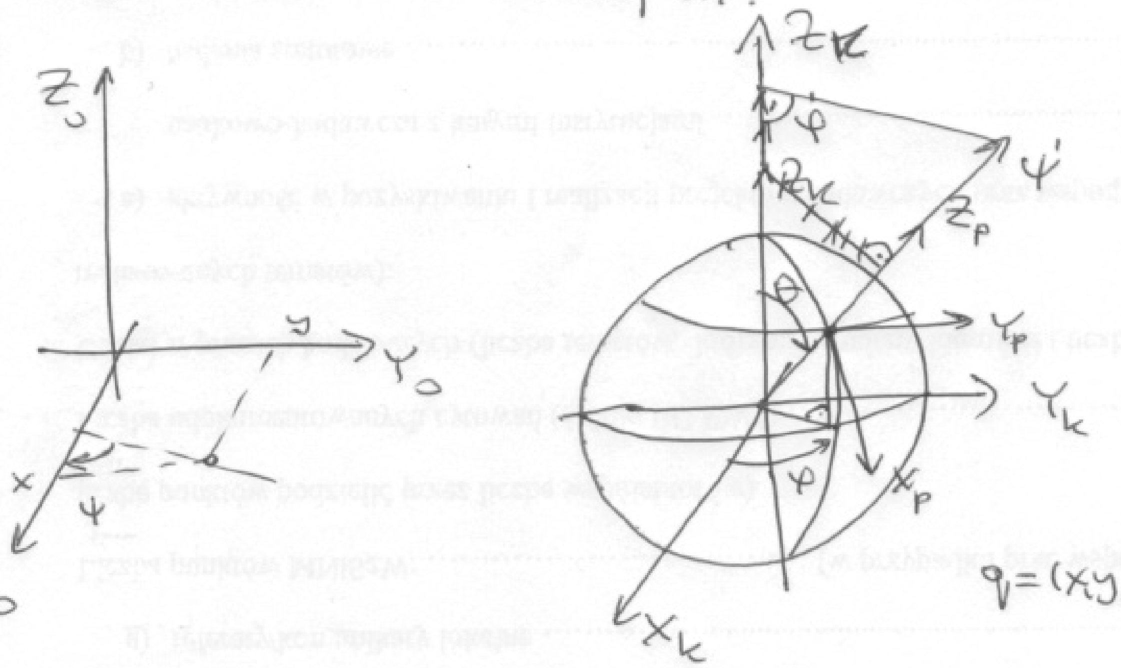
(2) $$\boxed{(mR^2+I_2)s_\theta\dot{\eta}_1 + \left((mR^2+I_2)s^2_\theta + I_{13}c^2_\theta\right)\dot{\eta}_2 + 2(mR^2+I_2-I_{13})\eta_2\eta_3 s_\theta c_\theta - I_2\eta_1\eta_3 c_\theta = 0}$$

$$mR^2(\cancel{\dot{\eta}_1 s_\varphi c_\varphi} - \eta_1\dot{\varphi} s^2_\varphi - \cancel{\dot{\eta}_1 s_\varphi c_\varphi} - \eta_1\dot{\varphi} c^2_\varphi)c_\theta + (I_{13}+mR^2)\dot{\eta}_3 +$$

$$+ (mR^2+I_2-I_{13})\eta_2^2 s_\theta c_\theta - I_2\eta_2\eta_1 c_\theta + mRg\, s_\theta = 0$$

(3) $$\boxed{(I_{13}+mR^2)\dot{\eta}_3 - mR^2\eta_1\eta_2 c_\theta + (mR^2+I_2-I_{13})\eta_2^2 s_\theta c_\theta - I_2\eta_1\eta_2 c_\theta + mgR\, s_\theta = 0}$$

Dynamika kuli toczącej się.



$$q=(x,y,\varphi,\theta,\psi)$$

Kula ma masę m i promień R i jest jednorodna. Układ kuli umieszczamy w środku kuli, który jest także środkiem masy kuli. Obliczamy lagranżian

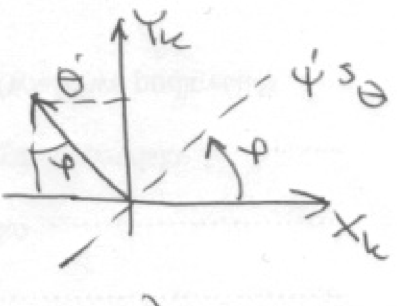
$$L = K - V$$

$$K = \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right) = \frac{1}{2} I\left(\omega_1^2+\omega_2^2+\omega_3^2\right) + \frac{1}{2}m(\dot{x}^2+\dot{y}^2), \quad I = \frac{2}{5}mR^2$$

$$\omega_1 = -\dot{\theta} s_\varphi + \dot{\psi} s_\theta c_\varphi$$

$$\omega_2 = \dot{\theta} c_\varphi + \dot{\psi} s_\theta s_\varphi$$

$$\omega_3 = \dot{\varphi} + \dot{\psi} c_\theta$$



$$L = K = \frac{1}{2} I\left(\dot{\varphi}^2+\dot{\theta}^2+\dot{\psi}^2\right) + I\dot{\varphi}\dot{\psi}c_\theta + \frac{1}{2}m(\dot{x}^2+\dot{y}^2), \quad V = \text{const} = 0$$

Ograniczenia

$$A(q)=\begin{bmatrix} 1 & 0 & -Rs_\theta s_\psi & -Rc_\psi & 0 \\ 0 & 1 & Rs_\theta c_\psi & -Rs_\psi & 0 \\ 0 & 0 & c_\theta & 0 & 1 \end{bmatrix}$$

$$G(q)=\begin{bmatrix} Rc_\psi & Rs_\psi s_\theta \\ Rs_\psi & -Rc_\psi s_\theta \\ 0 & 1 \\ 1 & 0 \\ 0 & -c_\theta \end{bmatrix} \qquad A^T=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -Rs_\theta s_\psi & Rs_\theta c_\psi & 0 \\ -Rc_\psi & -Rs_\psi & c_\theta \\ -Rc_\psi & -Rs_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Równanie dynamiki:

$$G^T=\begin{bmatrix} Rc_\psi & Rs_\psi & 0 & 1 & 0 \\ Rs_\psi s_\theta & -Rc_\psi s_\theta & 1 & 0 & -c_\theta \end{bmatrix}$$

$$\frac{\partial L}{\partial \dot x}=m\dot x,\ \frac{\partial L}{\partial \dot y}=m\dot y,\ \frac{\partial L}{\partial \dot\varphi}=I\dot\varphi+I\dot\psi c_\theta,\ \frac{\partial L}{\partial \dot\theta}=I\dot\theta,\ \frac{\partial L}{\partial \dot\psi}=I\dot\psi+I\dot\varphi c_\theta$$

$$\frac{\partial L}{\partial x}=\frac{\partial L}{\partial y}=\frac{\partial L}{\partial \varphi}=\frac{\partial L}{\partial \psi}=0,\ \frac{\partial L}{\partial \theta}=-I\dot\varphi\dot\psi s_\theta$$

$$m\ddot x=$$

$$m\ddot y= \qquad\qquad A^T\lambda \qquad \lambda=(\lambda_1,\lambda_2,\lambda_3)$$

$$I\ddot\varphi+I\ddot\psi c_\theta-I\dot\psi\dot\theta s_\theta=$$

$$I\ddot\theta+I\dot\varphi\dot\psi s_\theta=$$

$$I\ddot\psi+I\ddot\varphi c_\theta-I\dot\varphi\dot\theta s_\theta=$$

$$\begin{cases} mR\ddot x c_\psi+mR\ddot y s_\psi+I\ddot\theta+I\dot\varphi\dot\psi s_\theta=0 \\ mRs_\psi s_\theta\ddot x-mR\ddot y c_\psi s_\theta+I\ddot\varphi+\cancel{I\ddot\psi c_\theta}-I\dot\psi\dot\theta s_\theta-\cancel{I\ddot\psi c_\theta}-I\ddot\varphi c_\theta^2+I\dot\varphi\dot\theta s_\theta c_\theta=0 \end{cases}$$

$$\begin{cases} mR\ddot x c_\psi+mR\ddot y s_\psi+I\ddot\theta+I\dot\varphi\dot\psi s_\theta=0 \\ mR\ddot x s_\psi s_\theta-mR\ddot y c_\psi s_\theta+I\ddot\varphi s_\theta^2-I\dot\psi\dot\theta s_\theta+I\dot\varphi\dot\theta s_\theta c_\theta=0 \end{cases}$$

(9) $\boxed{\dot x=\eta_1 Rc_\psi+\eta_2 Rs_\psi s_\theta,\ \dot y=\eta_1 Rs_\psi-\eta_2 Rc_\psi s_\theta,\ \dot\varphi=\eta_2,\ \dot\theta=\eta_1,\ \dot\psi=-\eta_2 c_\theta}$

$$\ddot x=\dot\eta_1 Rc_\psi-\eta_1 R\dot\psi s_\psi+\dot\eta_2 Rs_\psi s_\theta+\eta_2 R\dot\psi c_\psi s_\theta+\eta_2 Rs_\psi\dot\theta c_\theta$$

$$\ddot y=\dot\eta_1 Rs_\psi+\eta_1 R\dot\psi c_\psi-\dot\eta_2 Rc_\psi s_\theta+\eta_2 R\dot\psi s_\psi s_\theta-\eta_2 Rc_\psi\dot\theta c_\theta$$

$$\ddot\varphi=\dot\eta_2,\ \ddot\theta=\dot\eta_1,\ \ddot\psi=-\dot\eta_2 c_\theta+\eta_2\dot\theta s_\theta$$

$$mR^2\dot\eta_1 c^2_\psi - mR^2\eta_1\dot\psi s_\psi c_\psi + mR^2\eta_2\dot\psi s_\psi s_\theta + mR^2\eta_2 R\dot\psi c_\psi s_\theta + mR^2\eta_2 R s_\psi c_\theta\dot\theta$$

$$+ mR^2\dot\eta_1 s^2_\psi + mR^2\eta_1\dot\psi s_\psi c_\psi - mR^2\dot\eta_2 s_\psi c_\psi s_\theta + mR^2\eta_2\dot\psi s^2_\psi s_\theta - mR^2\eta_2 s_\psi c_\psi\dot\theta c_\theta +$$

$$+ I\dot\eta_1 - I\eta_2^2 s_\theta c_\theta = 0$$

$$(mR^2+I)\dot\eta_1 + mR^2(-\eta_2^2 c_\theta s_\theta) - I\eta_2^2 s_\theta c_\theta = 0$$

$$(1)\quad \boxed{(mR^2+I)\dot\eta_1 - (mR^2+I)\eta_2^2 s_\theta c_\theta = 0}$$

$$mR^2\dot\eta_1 s_\psi c_\psi s_\theta - mR^2\eta_1\dot\psi s^2_\psi s_\theta + mR^2\dot\eta_2 s^2_\psi s^2_\theta + mR^2\eta_2\dot\psi s_\psi c_\psi s^2_\theta + mR^2\eta_2 s^2_\psi s_\theta c_\theta\dot\theta$$

$$- mR^2\dot\eta_1 s_\psi c_\psi s_\theta - mR^2\eta_1\dot\psi c^2_\psi s_\theta + mR^2\dot\eta_2 c^2_\psi s^2_\theta - mR^2\eta_2\dot\psi s_\psi c_\psi s^2_\theta + mR^2\eta_2 c^2_\psi s_\theta c_\theta\dot\theta +$$

$$+ I\dot\eta_2 s^2_\theta + I\eta_2\eta_1 s_\theta c_\theta + I\eta_1\eta_2 s_\theta c_\theta = 0$$

$$- mR^2\eta_1\dot\psi s_\theta + mR^2\dot\eta_2 s^2_\theta + mR^2\eta_2 s_\theta c_\theta\dot\theta + I\dot\eta_2 s^2_\theta + 2I\eta_1\eta_2 s_\theta c_\theta = 0$$

$$mR^2\eta_1\eta_2 s_\theta c_\theta + mR^2\dot\eta_2 s^2_\theta + mR^2\eta_2\eta_1 s_\theta c_\theta + I\dot\eta_2 s^2_\theta + 2I\eta_1\eta_2 s_\theta c_\theta = 0$$

$$(2)\quad \boxed{(mR^2+I)\dot\eta_2 s^2_\theta + 2(mR^2+I)\eta_1\eta_2 s_\theta c_\theta = 0}$$

(0), (1), (2) równania ruchu kuli toczącej się.

$$\begin{cases} \dot x = \eta_1 R c_\psi - \eta_2 R s_\psi s_\theta \\ \dot y = \eta_1 R s_\psi - \eta_2 R c_\psi s_\theta \\ \dot\varphi = \eta_2 \\ \dot\theta = \eta_1 \\ \dot\psi = -\eta_2 c_\theta \\ \dot\eta_1 - \eta_2^2 s_\theta c_\theta = 0 \\ \dot\eta_2 s^2_\theta + 2\eta_1\eta_2 s_\theta c_\theta = 0 \end{cases}$$