Mechanika analityczna, II r AiR

Wykład 10

Dynamika układów nieholonomicznych

1. Dynamika układów z ograniczeniami (nieholonomicznymi)

$q, \dot{q} \in \mathbb{R}^n$, $L(q, \dot{q})$, $A(q)\dot{q} = 0$, $\underset{l \times n}{A(q)}$, $\operatorname{rank} A(q) = l \leq n$

- piszemy równanie Lagrange'owskie układu swobodnego

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = F$$

- po prawej stronie umieszczamy siły przyczepności, które powodują spełnienie ograniczeń
- równanie mają postać

$$Q(q)\ddot{q} + P(q, \dot{q}) = F$$

- siły F wyznaczamy w oparciu o zasadę d'Alemberta: siły przyczepności nie wykonują pracy na dopuszczalnych przemieszczeniach

$$A(q)\dot{q} = 0 \Rightarrow \langle F, \dot{q} \rangle = F^T\dot{q} \stackrel{?}{=} 0$$

wynika stąd, że $F^T = \lambda^T A(q)$, czyli $F = A^T(q)\lambda$, $\lambda \in \mathbb{R}^l$ – wektor mnożników Lagrange'a

- podstawiamy

$$Q(q)\ddot{q} + P(q, \dot{q}) = A^T(q)\lambda \qquad (*)$$

- eliminujemy λ; jeżeli $A(q)\dot{q} = 0$, to istnieje $G(q)$, taka że $A(q)G(q) = 0$ i $\dot{q} = G(q)\eta$, zatem $G^T A^T = 0$,

$$G^T Q\ddot{q} + G^T P = 0$$

- podstawiamy $\ddot{q} = \dot{G}(q)\eta + G(q)\dot{\eta}$ i otrzymujemy

$$G^T Q G\dot{\eta} + G^T Q\dot{G}\eta + G^T P = 0,$$

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– G^TQG jest odwracalne, zatem ostatecznie

$$\begin{cases} \dot\eta = -(G^TQG)^{-1}G^T(Q\dot G\eta + P) \\ \dot q = G(q)\eta \end{cases}$$

– równanie ruchu

– jeśli interesują nas siły przyczepności, mnożymy (*) przez $A(q)$

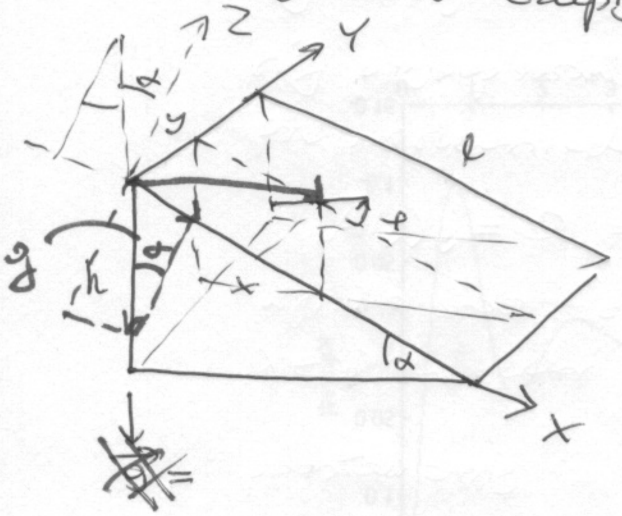
$$AQ\ddot q + AP = AA^T\lambda$$

$$\lambda = (AA^T)^{-1}A(Q\ddot q + P), \quad F = A^T(AA^T)^{-1}A(Q\ddot q + P)$$

Mając siły tarcia T możemy określić, przy jakich przyspieszeniach $\ddot q$ układ wpadnie w poślizg: $F > T$.

Przykłady:

Ex1 Łyżwiarz Czapłygina (Čaplygin)



$$q = (x, y, \varphi)$$

$$A(q) = [s\varphi, -c\varphi, 0]$$

$$K = \frac{1}{2}\dot x^2 + \frac{1}{2}\dot y^2 + \frac{1}{2}\dot\varphi^2$$

$$V = -m\langle \vec g, \vec r\rangle$$

$$\vec g = (+gs_\alpha, 0, *) \quad \vec r = (x, y, 0), \quad s_\alpha = \frac{h}{l}$$

$$V = -mgxs_\alpha = -mg\frac{h}{l}x \cong -x$$

$$L = \frac{1}{2}\dot x^2 + \frac{1}{2}\dot y^2 + \frac{1}{2}\dot\varphi^2 + x$$

Równanie dynamiki:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q} - \frac{\partial L}{\partial q} = \Phi = A^T\lambda \quad , \lambda \in \mathbb{R}$$

$$\frac{\partial L}{\partial \dot x} = \dot x \ , \frac{\partial L}{\partial \dot y} = \dot y \ , \frac{\partial L}{\partial \dot\varphi} = \dot\varphi \ , \frac{\partial L}{\partial x} = 1 \ , \frac{\partial L}{\partial y} = \frac{\partial L}{\partial \varphi} = 0$$

$$\begin{matrix} \ddot x - 1 \\ \ddot y \\ \ddot\varphi \end{matrix} = A^T\lambda = \begin{pmatrix} \lambda s_\varphi \\ -\lambda c_\varphi \\ 0 \end{pmatrix}$$

$$\begin{cases} \ddot x - 1 = \lambda s_\varphi & | c_\varphi \\ \ddot y = -\lambda c_\varphi & | s_\varphi \\ \ddot\varphi = 0 \end{cases} \qquad G = \begin{bmatrix} c_\varphi & 0 \\ s_\varphi & 0 \\ 0 & 1 \end{bmatrix}$$

$$\dot x = \eta_1 c_\varphi \ , \dot y = \eta_1 s_\varphi \ , \dot\varphi = \eta_2$$

$$\ddot x c_\varphi - c_\varphi + \ddot y s_\varphi = 0 \qquad \ddot x = \dot\eta_1 c_\varphi - \eta_1 \dot\varphi s_\varphi$$

$$\ddot\varphi = 0 \qquad \ddot y = \dot\eta_1 s_\varphi + \eta_1 \dot\varphi c_\varphi$$

$$\dot\eta_1 c^2_\varphi - \cancel{\eta_1 \dot\varphi s_\varphi c_\varphi} - c_\varphi + \dot\eta_1 s^2_\varphi + \cancel{\eta_1 \dot\varphi s_\varphi c_\varphi} = 0$$

$$\begin{cases} \dot\eta_1 = c_\varphi \\ \dot\eta_2 = 0 \end{cases} \Rightarrow \eta_2 = const = \dot\varphi = \omega \Rightarrow \varphi = \omega t \ , \varphi(0) = 0$$

$$\dot\eta = \cos\omega t$$

$$\eta - \eta_0 = \frac{1}{\omega}\sin\omega t \Big|_0^t \ , \eta(0) = 0$$

$$\boxed{\eta(t) = \frac{1}{\omega}\sin\omega t}$$

$$\dot x = \frac{1}{\omega}\sin\omega t\cos\omega t = \frac{1}{2\omega}\sin 2\omega t$$

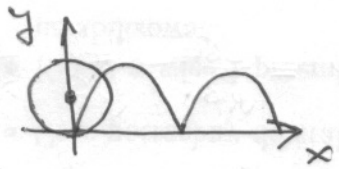
$$\dot y = \frac{1}{\omega}\sin^2\omega t = \frac{1}{2\omega}(1-\cos 2\omega t)$$

Niech $x(0) = y(0) = 0$, $\varphi(0) = 0$, $\omega = \dot\varphi(0) \neq 0$

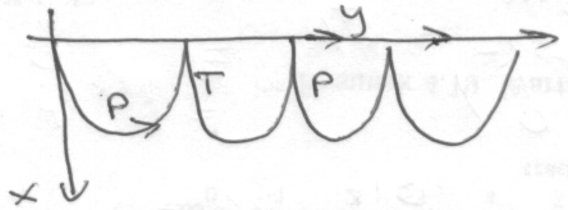
Mamy $x(t) - x(0) = -\frac{1}{4\omega^2}\cos 2\omega t\Big|_0^t = \frac{1}{4\omega^2}(1-\cos 2\omega t)$

$$y(t) - y(0) = \frac{1}{2\omega}\left(t - \frac{1}{2\omega}\sin 2\omega t\right)\Big|_0^t = \frac{1}{4\omega^2}(2\omega t - \sin 2\omega t)$$

$x(t) = a(t - \sin t)$, $y(t) = a(1 - \cos t)$ – cykloida



Mamy $a \simeq \frac{1}{4\omega^2}$, $t \simeq 2\omega t$, $x \Leftrightarrow y$



Tyzwian Grapiyyina jedzie po cykloidzie.

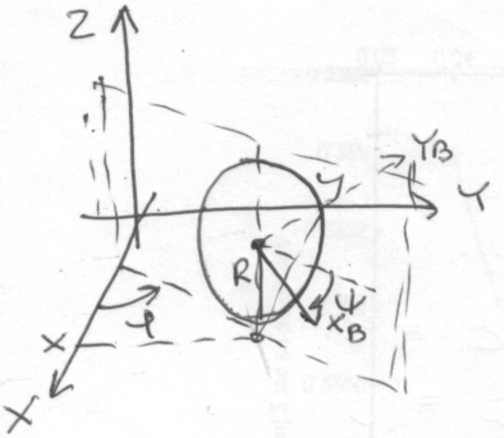
Siły przyczepności:

$$\ddot{x} - 1 = \lambda s_\varphi \quad | s_\varphi$$
$$\ddot{y} = -\lambda c_\varphi \quad | -c_\varphi$$

$$\ddot{x} s_\varphi - s_\varphi - \ddot{y} c_\varphi = \lambda$$

$$F = A^T \lambda = \begin{pmatrix} s_\varphi \\ -c_\varphi \\ 0 \end{pmatrix} (\ddot{x} s_\varphi - \ddot{y} c_\varphi - s_\varphi)$$

Ex2: Koło toczące się pionowo



$$q = (x, y, \varphi, \psi)$$

$$A(q) = \begin{bmatrix} s_\varphi & -c_\varphi & 0 & 0 \\ c_\varphi & s_\varphi & 0 & -R \end{bmatrix}$$

$$L = K - V, \quad V = \text{const}$$

$$L = K = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2} I_1 \dot{\varphi}^2 + \frac{1}{2} I_2 \dot{\psi}^2$$

$$I_1 = \frac{1}{4} mR^2, \quad I_2 = \frac{1}{2} mR^2$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = F = A^T \lambda, \quad \lambda \in \mathbb{R}^2$$

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$$A^T = \begin{bmatrix} s_\varphi & c_\varphi \\ -c_\varphi & s_\varphi \\ 0 & 0 \\ 0 & -R \end{bmatrix}$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x},\ \frac{\partial L}{\partial \dot{y}} = m\dot{y},\ \frac{\partial L}{\partial \dot{\varphi}} = I_1\dot{\varphi},\ \frac{\partial L}{\partial \dot{\psi}} = I_2\dot{\psi},\ \frac{\partial L}{\partial x} = \frac{\partial L}{\partial y} = \frac{\partial L}{\partial \varphi} = \frac{\partial L}{\partial \psi} = 0$$

$$\begin{pmatrix} m\ddot{x} \\ m\ddot{y} \\ I_1\ddot{\varphi} \\ I_2\ddot{\psi} \end{pmatrix} = A^T\lambda = \begin{pmatrix} \lambda_1 s_\varphi + \lambda_2 c_\varphi \\ -\lambda_1 c_\varphi + \lambda_2 s_\varphi \\ 0 \\ -\lambda_2 R \end{pmatrix}$$

$$AG = 0 \Rightarrow G = \begin{bmatrix} Rc_\varphi & 0 \\ Rs_\varphi & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad G^T = \begin{bmatrix} Rc_\varphi & Rs_\varphi & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$G^T \Big| \Rightarrow \begin{aligned} & mR\ddot{x}c_\varphi + mR\ddot{y}s_\varphi + I_2\ddot{\psi} = 0 \\ & I_1\ddot{\varphi} = 0 \Rightarrow \dot{\varphi} = \text{const} - \omega,\ \varphi = \omega t \\ & \varphi(0) = 0 \end{aligned}$$

$$\dot{x} = \eta_1 Rc_\varphi,\ \dot{y} = \eta_1 Rs_\varphi,\ \dot{\varphi} = \eta_2,\ \dot{\psi} = \eta_1$$

$$\ddot{x} = \dot{\eta}_1 Rc_\varphi - \eta_1 R\dot{\varphi}s_\varphi,\ \ddot{y} = \dot{\eta}_1 Rs_\varphi + \eta_1 R\dot{\varphi}c_\varphi,\ \ddot{\psi} = \dot{\eta}_1,\ \eta_2 = \omega$$

$$mR^2\dot{\eta}_1 c^2_\varphi - \cancel{mR^2\dot{\varphi}s_\varphi c_\varphi} + mR\dot{\eta}_1 R^2 s^2_\varphi + \cancel{mR^2\dot{\varphi}s_\varphi c_\varphi} + I_2\dot{\eta}_1 = 0$$

$$mR^2\dot{\eta}_1 + I_2\dot{\eta}_1 = (mR^2 + \cancel{\tfrac{1}{2}mR^2}\ I_2)\dot{\eta}_1 = 0 \Rightarrow \eta_1 = \text{const} = \eta = \dot{\psi}(0)$$

Otrzymujemy: $\dot{x} = \eta R\cos\omega t \Rightarrow x(t) - x(0) = \frac{\eta R}{\omega}\sin\omega t\Big|_0^t$

$x(0) = 0 \Rightarrow x(t) = \frac{\eta R}{\omega}\sin\omega t$

$\dot{y} = \eta R\sin\omega t \Rightarrow y(t) - y(0) = -\frac{\eta R}{\omega}\cos\omega t\Big|_0^t =$

$y(0) = 0 \Rightarrow y(t) = \frac{\eta R}{\omega}(1 - \cos\omega t)$

Jaki to ruch?

$$x^2+\left(y-\frac{\eta R}{\omega}\right)^2=\left(\frac{\eta R}{\omega}\right)^2$$